# *OTHER EQUATION TYPES*

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# 1.6
# OTHER TYPES OF EQUATIONS

- Rational Equations
- Equations with Radicals
- Equations with Rational Exponents
- Equations Quadratic in Form

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# *Rational Equations*


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# 1.6
# OTHER TYPES OF EQUATIONS

## RATIONAL EQUATIONS

A ***rational equation*** is an equation that has a rational expression for one or more terms.

To solve a rational equation, multiply each side by the least common denominator (LCD) of all the terms and then solve the resulting equation.

Because a rational expression is not defined when its denominator is 0, ***proposed solutions for which any denominator equals 0 are excluded from the solution set***.

Example 1

# Solving Rational Equations that Lead to Linear Equations

**1(a)** Solve the equation: $$\frac{3x-1}{3}-\frac{2x}{x-1}=x$$

*Solution:*

The least common denominator is $3(x-1)$, which is equal to $0$ if $x=1$. Therefore, 1 cannot possibly be a solution of this equation.

Multiply by the LCD, $3(x-1)$, where $x \neq 1$.

$$3(x-1)\left(\frac{3x-1}{3}\right)-3(x-1)\left(\frac{2x}{x-1}\right)=3(x-1)x$$

$$(x-1)(3x-1)-3(2x)=3x(x-1)$$

Divide out common factors.

Example 1

# Solving Rational Equations that Lead to Linear Equations

*Solution (cont'd):*

$$(x-1)(3x-1)-3(2x)=3x(x-1)$$

$$3x^2-4x+1-6x=3x^2-3x$$ Multiply.

$$1-10x=-3x$$ Subtract $3x^2$ *(both sides).* Combine like terms.

$$1=7x$$ Solve the linear equation.

$$x=\frac{1}{7}$$ Proposed solution.

The proposed solution meets the requirement that $x \neq 1$ and does not cause any denominator to equal 0.
Substitute to check for correct algebra …

The solution set is $\left\{\frac{1}{7}\right\}$

Example 1

# Solving Rational Equations that Lead to Linear Equations

**1(b)** Solve the equation: $$\frac{x}{x-2}=\frac{2}{x-2}+2$$

*Solution:*

Multiply by the LCD, $x-2$, where $x \neq 2$.

$$(x-2)\left(\frac{x}{x-2}\right)=(x-2)\left(\frac{2}{x-2}\right)+(x-2)2$$

$$x=2+2(x-2)$$ Divide out common factors.

$$x=2+2x-4$$ Distributive property.

$$-x=-2$$ Solve the linear equation.

$$x=2$$ Proposed solution.

Example 1

# Solving Rational Equations that Lead to Linear Equations

**1(b)** Solve the equation: $\frac{x}{x-2} = \frac{2}{x-2} + 2$

*Solution (cont'd):*

$$x = 2$$ Proposed solution.

The proposed solution is 2.
However, the variable is restricted to real numbers except 2.
If $x = 2$, then not only does it cause a zero denominator,
but multiplying by $x - 2$ in the first step is multiplying both sides by $0$,
which is not valid. Thus, …

The solution set is $\emptyset$ .

## Example 2

# Solving Rational Equations that Lead to Quadratic Equations

**2(a)** Solve the equation: $\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$

*Solution:*

$$\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x(x-2)}$$

Factor the last denominator.

Multiply by $x(x-2)$, where $x \neq 0,\ 2$.

$$x(x-2)\left(\frac{3x+2}{x-2}\right)+x(x-2)\left(\frac{1}{x}\right)=x(x-2)\left(\frac{-2}{x(x-2)}\right)$$

$$x(3x+2)+(x-2)=-2$$

Divide out common factors.

## Example 2
# Solving Rational Equations that Lead to Quadratic Equations

**2(a)** Solve the equation: $$\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$$

*Solution (cont'd):*

$$x(3x+2)+(x-2)=-2$$

$$3x^2+2x+x-2=-2$$ Distributive property.

$$3x^2+3x=0$$ Standard form.

$$3x(x+1)=0$$ Factor.

$$3x=0 \text{ or } x+1=0$$ Zero-factor property.

**Set each factor equal to 0.**

## Example 2
# Solving Rational Equations that Lead to Quadratic Equations

**2(a)** Solve the equation: $$\frac{3x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^2-2x}$$

*Solution (cont'd):*

$$3x = 0 \text{ or } x + 1 = 0$$

$$x = 0 \text{ or } x = -1$$ Proposed solutions.

Because of the restriction $x \neq 0$, the only valid solution is $-1$ …

The solution set is $\{-1\}$

## Example 2

# Solving Rational Equations that Lead to Quadratic Equations

**2(b)** Solve the equation: $$\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^2-1}$$

*Solution:*

$$\frac{-4x}{x-1}+\frac{4}{x+1}=\frac{-8}{(x+1)(x-1)}$$

Factor.

$$(x+1)(x-1)\left(\frac{-4x}{x-1}\right)+(x+1)(x-1)\frac{4}{x+1}$$

$$=(x+1)(x-1)\left(\frac{-8}{(x+1)(x-1)}\right)$$

Multiply by $(x+1)(x-1)$, $x \neq \pm 1$.

Example 2

# Solving Rational Equations that Lead to Quadratic Equations

**2(b)** Solve the equation:

*Solution (cont'd):*

| | |
|---|---|
| $-4x(x+1)+4(x-1)=-8$ | Divide out common factors. |
| $-4x^2-4x+4x-4=-8$ | Distributive property. |
| $-4x^2+4=0$ | Standard form. |
| $x^2-1=0$ | Divide by $-4$. |
| $(x+1)(x-1)=0$ | Factor. |

Example 2

# Solving Rational Equations that Lead to Quadratic Equations

**2(b)** Solve the equation: $\dfrac{-4x}{x-1} + \dfrac{4}{x+1} = \dfrac{-8}{x^2-1}$

*Solution (cont'd):*

$$(x+1)(x-1) = 0$$

$x + 1 = 0$ or $x - 1 = 0$ — Zero-factor Property.

$x = -1$ or $x = 1$ — Proposed solutions.

Neither proposed solution is valid …

The solution set is $\emptyset$

# Equations Involving Radicals

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# POWER PROPERTY=

## POWER PROPERTY

If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are algebraic expressions,
then every solution of the equation

$$\boldsymbol{P} = \boldsymbol{Q}$$

is also a solution of the equation

$$\boldsymbol{P}^{\boldsymbol{n}} = \boldsymbol{Q}^{\boldsymbol{n}}$$

for any positive integer $\boldsymbol{n}$.

# POWER PROPERTY

## Caution

***Be very careful when using the power property***.

It does *not* say that the equations

$\boldsymbol{P = Q}$ and $\boldsymbol{P^n = Q^n}$

are equivalent;

it says only that each solution of the original equation

$\boldsymbol{P = Q}$

is also a solution of the new equation

$\boldsymbol{P^n = Q^n}$.

# SOLVING an EQUATION INVOLVING RADICALS

## Solving an Equation Involving Radicals

***Step 1*** Isolate the radical on one side of the equation.

***Step 2*** Raise each side of the equation to a power that is the same as the index of the radical so that the radical is eliminated.

***If the equation still contains a radical, repeat Steps 1 and 2.***

***Step 3*** Solve the resulting equation.

***Step 4*** Check each proposed solution in the *original* equation.

Example 3

# Solving an Equation Containing a Radical (Square Root)

**3(a)** Solve: $x - \sqrt{15 - 2x} = 0$

*Solution:*

$$x = \sqrt{15 - 2x}$$ Isolate the radical.

$$x^2 = \left(\sqrt{15 - 2x}\right)^2$$ Square each side.

$$x^2 = 15 - 2x$$

$$x^2 + 2x - 15 = 0$$ Solve the quadratic equation.

$$(x + 5)(x - 3) = 0$$

$x + 5 = 0$ or $x - 3 = 0$ Zero-factor property.

$x = -5$ or $x = 3$ Proposed solutions.

Only 3 is a valid solution …

The solution set is $\{3\}$

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## Example 4

# Solving an Equation Containing Two Radicals

**4(a)** Solve: $\sqrt{2x+3} - \sqrt{x+1} = 1$

*Solution:*

When an equation contains two radicals,
begin by isolating one of the radicals on one side of the equation.

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$

$$\sqrt{2x+3} = 1 + \sqrt{x+1}$$ Isolate $\sqrt{2x+3}$.

$$\left(\sqrt{2x+3}\right)^2 = \left(1 + \sqrt{x+1}\right)^2$$ Square each side.

Example 4

# Solving an Equation Containing Two Radicals

**4(a)** Solve: $\sqrt{2x+3} - \sqrt{x+1} = 1$

*Solution (cont'd):*

$$\left(\sqrt{2x+3}\right)^2 = \left(1 + \sqrt{x+1}\right)^2$$

$$2x + 3 = 1 + 2\sqrt{x+1} + (x+1)$$ Be careful !

**Don't forget this term when squaring.**

$$x + 1 = 2\sqrt{x+1}$$ Isolate the remaining radical.

$$(x+1)^2 = \left(2\sqrt{x+1}\right)^2$$ Square again.

$$x^2 + 2x + 1 = 4(x+1)$$ Apply the exponents.

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### Example 4

# Solving an Equation Containing Two Radicals

**4(a)** Solve: $\sqrt{2x+3} - \sqrt{x+1} = 1$

*Solution (cont'd):*

| | |
|---|---|
| $x^2 + 2x + 1 = 4x + 4$ | Distributive property. |
| $x^2 - 2x - 3 = 0$ | Solve the quadratic equation. |
| $(x-3)(x+1) = 0$ | Factor. |
| $x - 3 = 0$ or $x + 1 = 0$ | Zero-factor property. |
| $x = 3$ or $x = -1$ | Proposed solutions. |

Check each proposed solution in the *original* equation.

Both $3$ and $-1$ are solutions of the original equation …

The solution set is $\{-1, 3\}$

# Solving an Equation Containing Two Radicals

## Caution

Remember to isolate a radical in ***Step 1***.

It would be incorrect to square each term individually as the first step in **Example 4**.

Example 5

# Solving an Equation Containing a Radical (Cube Root)

**5(a)** Solve: $\sqrt[3]{4x^2 - 4x + 1} - \sqrt[3]{x} = 0$

*Solution:*

$$\sqrt[3]{4x^2 - 4x + 1} = \sqrt[3]{x}$$ Isolate a radical.

$$\left(\sqrt[3]{4x^2 - 4x + 1}\right)^3 = \left(\sqrt[3]{x}\right)^3$$ Cube each side.

$$4x^2 - 4x + 1 = x$$ Apply the exponents.

$$4x^2 - 5x + 1 = 0$$ Solve the quadratic equation.

$$(4x - 1)(x - 1) = 0$$ Factor.

## Example 5

# Solving an Equation Containing a Radical (Cube Root)

**5(a)** Solve: $\sqrt[3]{4x^2 - 4x + 1} - \sqrt[3]{x} = 0$

*Solution (cont'd):*

$$(4x - 1)(x - 1) = 0$$

$4x - 1 = 0$ or $x - 1 = 0$ Zero-factor property.

$x = \frac{1}{4}$ or $x = 1$ Proposed solutions

Both are valid solutions …

The solution set is $\left\{\frac{1}{4}, 1\right\}$

# *Equations with Rational Exponents*

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## Example 6

# Solving Equations with Rational Exponents

**6(a)** Solve: $x^{3/5} = 27$

*Solution:*

$$x^{3/5} = 27$$

$$\left(x^{3/5}\right)^{5/3} = 27^{5/3}$$

Raise each side to the power $5/3$, the reciprocal of the exponent of $x$.

$$x = 243$$

$$27^{5/3} = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$$

The solution set is $\{\mathbf{243}\}$

Example 6

# Solving Equations with Rational Exponents

**6(b)** Solve: $(x-4)^{2/3}=16$

*Solution:*

$$\left[(x-4)^{2/3}\right]^{3/2}=\pm 16^{3/2}$$

Raise each side to the power $3/2$. Insert $\pm$ since this involves an even root, as indicated by the 2 in the denominator.

$$x-4=\pm 64$$

$\pm 16^{3/2}=\pm\left(\sqrt{16}\right)^3=\pm 4^3=\pm 64$

$$x=4\pm 64$$

$x=-60$ or $x=68$ Proposed solutions.

Both proposed solutions check in the original equation …

The solution set is $\{-60, 68\}$

# EQUATIONS QUADRATIC in FORM

## Equations Quadratic in Form

An equation is said to be ***quadratic in form***
if it can be written as

$$au^2 + bu + c = 0,$$

Where $a \neq 0$ and $u$ is some algebraic expression.

## Example 7

# Solving Equations Quadratic in Form

**7(a)** Solve: $(x+1)^{2/3}-(x+1)^{1/3}-2=0$

*Solution:*

Since $(x+1)^{2/3}=\left[(x+1)^{1/3}\right]^2$ Let $\boldsymbol{u}=(\boldsymbol{x}+\mathbf{1})^{\mathbf{1/3}}$.

$$u^2-u-2=0$$ Substitute.

$$(u-2)(u+1)=0$$ Factor.

$u-2=0$ or $u+1=0$ Zero-factor property

$u=2$ or $u=-1$

Example 7

# Solving Equations Quadratic in Form

**7(a)** Solve: $(x+1)^{2/3}-(x+1)^{1/3}-2=0$

*Solution (cont'd):*

**Don't forget this step.**

$u=2$ or $u=-1$

$(x+1)^{1/3}=2$ or $(x+1)^{1/3}=-1$ Replace $\boldsymbol{u}$ with $(\boldsymbol{x}+\mathbf{1})^{\mathbf{1/3}}$.

$\left[(x+1)^{1/3}\right]^3=2^3$ or $\left[(x+1)^{1/3}\right]^3=(-1)^3$ Cube each side.

$x=7$ or $x=-2$ Proposed solutions.

Both proposed solutions check in the original equation …

The solution set is $\{-2,7\}$

## Example 7
# Solving Equations Quadratic in Form

**7(b)** Solve: $6x^{-2} + x^{-1} = 2$

*Solution:*

$6x^{-2} + x^{-1} - 2 = 0$ — Subtract 2 *(both sides)*.

$6u^2 + u - 2 = 0$ — Let $u = x^{-1}$; then $u^2 = x^{-2}$.

$(3u + 2)(2u - 1) = 0$ — Factor.

$3u + 2 = 0 \text{ or } 2u - 1 = 0$ — Zero-factor property.

$u = -\frac{2}{3} \text{ or } u = \frac{1}{2}$

**Remember to substitute for *u*.**

## Example 7
# Solving Equations Quadratic in Form

**7(b)** Solve: $6x^{-2} + x^{-1} = 2$

*Solution (cont'd):*

$$u = -\frac{2}{3} \text{ or } u = \frac{1}{2}$$

$$x^{-1} = -\frac{2}{3} \text{ or } x^{-1} = \frac{1}{2}$$ Resubstitute.

$$x = -\frac{3}{2} \text{ or } x = 2$$ $x^{-1}$ is the reciprocal of $x$

Both proposed solutions check in the original equation …

The solution set is $\left\{-\frac{3}{2},\ 2\right\}$

## Caution

***When using a substitution variable in solving an equation that is quadratic in form, do not forget the step that gives the solution in terms of the original variable.***

## Example 8

# Solving Equations Quadratic in Form

**8(a)** Solve: $12x^4 - 11x^2 + 2 = 0$

*Solution:*

$12(x^2)^2 - 11x^2 + 2 = 0$ — $x^4 = (x^2)^2$

$12u^2 - 11u + 2 = 0$ — Let $\boldsymbol{u} = \boldsymbol{x^2}$; then $\boldsymbol{u^2} = \boldsymbol{x^4}$.

$(3u - 2)(4u - 1) = 0$ — Solve the quadratic equation.

$3u - 2 = 0$ or $4u - 1 = 0$ — Zero-property factor.

$u = \frac{2}{3}$ or $u = \frac{1}{4}$ — Solve the linear equations.

# Solving Equations Quadratic in Form

**8(a)** Solve: $12x^4 - 11x^2 + 2 = 0$

*Solution (cont'd):*

$$u = \frac{2}{3} \text{ or } u = \frac{1}{4}$$

$$x^2 = \frac{2}{3} \text{ or } x^2 = \frac{1}{4}$$ Replace $\boldsymbol{u}$ with $\boldsymbol{x^2}$.

$$x = \pm\sqrt{\frac{2}{3}} \text{ or } x = \pm\sqrt{\frac{1}{4}}$$ Square root property.

## Example 8
# Solving Equations Quadratic in Form

**8(a)** Solve: $12x^4 - 11x^2 + 2 = 0$

*Solution (cont'd):*

$$x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \text{ or } x = \pm\frac{1}{2}$$

Simplify radicals.

$$x = \pm\frac{\sqrt{6}}{3} \text{ or } x = \pm\frac{1}{2}$$

The solution set is $\left\{\pm\frac{\sqrt{6}}{3}, \pm\frac{1}{2}\right\}$

# Solving Equations Quadratic in Form

## Note

Some equations that are quadratic in form
are simple enough to avoid using
the substitution variable technique. To solve …

$$12x^4 - 11x^2 + 2 = 0$$

we could factor directly as $(3x^2 - 2)(4x^2 - 1)$,
set each factor equal to zero, and
then solve the resulting two quadratic equations.
***Which method to use is a matter of personal preference.***