QUADRATIC EQUATIONS

Terrametra Resources
Lynn Patten

1.4
QUADRATIC EQUATIONS

- The Zero-Factor Property
- The Square Root Property
- Completing the Square
- The Quadratic Formula
- Solving for a Specified Variable
- The Discriminant

QUADRATIC EQUATIONS

(Second Degree Equations)

QUADRATIC EQUATIONS

A ***quadratic equation*** is a ***second-degree equation***, that is, an equation with a squared variable term and no terms of greater degree.

$x^2 = 25 \qquad 4x^2 + 4x - 5 = 0 \qquad 3x^2 = 4x - 8$

ZERO-FACTOR PROPERTY

ZERO-FACTOR PROPERTY

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are complex numbers with $\boldsymbol{ab = 0}$,

then $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$ or **both** equal zero.

Example 1

Using the Zero-Factor Property

1(a) Solve: $6x^2 + 7x = 3$

Solution:

$$6x^2 + 7x - 3 = 0$$ Standard form.

$$(3x - 1)(2x + 3) = 0$$ Factor.

$$3x - 1 = 0 \text{ or } 2x + 3 = 0$$ Zero-factor property.

$$3x = 1 \text{ or } 2x = -3$$ Solve each equation.

$$x = \frac{1}{3} \text{ or } x = -\frac{3}{2}$$

SQUARE ROOT PROPERTY

SQUARE ROOT PROPERTY

If $x^2 = k$, then

$x = \sqrt{k}$ or $x = -\sqrt{k}$

SQUARE ROOT PROPERTY

A quadratic equation of the form $\boldsymbol{x^2 = k}$ can be solved by factoring.

$x^2 - k = 0$	Subtract $\boldsymbol{k}$ *(both sides)*.
$\left(x - \sqrt{k}\right)\left(x + \sqrt{k}\right) = 0$	Factor.
$x - \sqrt{k} = 0 \text{ or } x + \sqrt{k} = 0$	Zero-factor property.
$x = \sqrt{k} \text{ or } x = -\sqrt{k}$	Solve each equation.

SQUARE ROOT PROPERTY

That is, the solution set of $\boldsymbol{x^2 = k}$ is $\{\sqrt{\boldsymbol{k}}, -\sqrt{\boldsymbol{k}}\}$ which may be abbreviated $\{\pm\sqrt{\boldsymbol{k}}\}$.

Both solutions are real if $\boldsymbol{k > 0}$.
Both are pure imaginary if $\boldsymbol{k < 0}$.
If $\boldsymbol{k < 0}$, we write the solution set as $\left\{\pm i\sqrt{|k|}\right\}$

If $\boldsymbol{k = 0}$, then there is only one distinct solution, $\boldsymbol{0}$, sometimes called a ***double solution***.

Example 2

Using the Square Root Property

2(a) Solve: $x^2 = 17$

Solution:

By the square root property …

The solution set of $x^2 = 17$ is $\{\pm\sqrt{\mathbf{17}}\}$.

Example 2

Using the Square Root Property

2(b) Solve: $x^2 = -25$

Solution:

By the square root property and since $\sqrt{-1} = i$ ...

The solution set of $x^2 = -25$ is $\{\pm \boldsymbol{5i}\}$.

TERRAMETRA

Example 2

Using the Square Root Property

2(c) Solve: $(x-4)^2 = 12$

Solution:

$$x - 4 = \pm\sqrt{12}$$

Generalized square root property.

$$x = 4 \pm \sqrt{12}$$

Add 4 *(both sides).*

$$x = 4 \pm 2\sqrt{3}$$

$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

The solution set is $\{\mathbf{4 \pm 2\sqrt{3}}\}$.

COMPLETING the SQUARE

COMPLETING the SQUARE

To solve $ax^2 + bx + c = 0$, where $a \neq 0$,
by completing the square, use these steps:

Step 1 If $a \neq 1$, divide both sides of the equation by a.

Step 2 Rewrite the equation so that the constant term is alone on one side of the equality symbol.

Step 3 Square half the coefficient of x, and add this square to each side of the equation.

Step 4 Factor the resulting trinomial as a perfect square and combine like terms on the other side.

Step 5 Use the square root property to complete the solution.

Example 3
Completing the Square
($a = 1$)

3(a) Solve: $x^2 - 4x - 14 = 0$

Solution:

Step 1 This step is not necessary since $a = 1$.

Step 2 $x^2 - 4x = 14$ Add 14 *(both sides)*.

Step 3 $x^2 - 4x + 4 = 14 + 4$ $\left[\frac{1}{2}(-4)\right]^2 = 4$ Add 4 *(both sides)*.

Step 4 $(x - 2)^2 = 18$ Factor; Combine like terms.

3(a) Solve: $x^2 - 4x - 24 = 0$

Solution (cont'd):

Step 5 $x - 2 = \pm\sqrt{18}$ Square root property.

Take *both* roots.

$x = 2 \pm \sqrt{18}$ Add 2 *(both sides)*.

$x = 2 \pm 3\sqrt{2}$ Simplify the radical.

The solution set is $\{2 \pm 3\sqrt{2}\}$.

Example 4

Completing the Square

($a \neq 1$)

4(a) Solve: $9x^2 - 12x + 9 = 0$

Solution:

$$9x^2 - 12x + 9 = 0$$

Step 1 $x^2 - \frac{4}{3}x + 1 = 0$ — Divide by 9 (*both sides*).

Step 2 $x^2 - \frac{4}{3}x = -1$ — Subtract 1 (*both sides*).

$$\left[\frac{1}{2}\left(-\frac{4}{3}\right)\right]^2 = \frac{4}{9}$$

Step 3 $x^2 - \frac{4}{3}x + \frac{4}{9} = -1 + \frac{4}{9}$ — Add $\frac{4}{9}$ (*both sides*).

Step 4 $\left(x - \frac{2}{3}\right)^2 = -\frac{5}{9}$ — Factor; Combine like terms.

4(a) Solve: $9x^2 - 12x + 9 = 0$

Solution (cont'd):

$$\left(x - \frac{2}{3}\right)^2 = -\frac{5}{9}$$

Step 5

$$x - \frac{2}{3} = \pm\sqrt{-\frac{5}{9}}$$

Square root property.

$$x - \frac{2}{3} = \pm\frac{\sqrt{5}}{3}i$$

$$\sqrt{-\frac{5}{9}} = \frac{\sqrt{-5}}{\sqrt{9}} = \frac{i\sqrt{5}}{3} \text{ or } \frac{\sqrt{5}}{3}i$$

$$x = \frac{2}{3} \pm \frac{\sqrt{5}}{3}i$$

Add $\frac{2}{3}$ (*both sides*).

The solution set is $\left\{\frac{2}{3} \pm \frac{\sqrt{5}}{3}i\right\}$.

QUADRATIC FORMULA

QUADRATIC FORMULA

The solutions of the ***quadratic equation***

$ax^2 + bx + c = 0$, where $a \neq 0$,

are given by the ***quadratic formula*** …

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

QUADRATIC FORMULA

That is, if we start with the equation …
$ax^2 + bx + c = 0$, for $a > 0$,
and complete the square to solve for x
in terms of the constants a, b, and c,
the result is a general formula
for solving any quadratic equation.

QUADRATIC FORMULA

Caution

Remember the fraction bar in the quadratic formula extends under the $-b$ term in the numerator.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 5

Using the Quadratic Formula

(Real Solutions)

5(a) Solve: $x^2 - 4x = -2$

Solution:

$$x^2 - 4x + 2 = 0$$

Standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula.

$a = 1,\ b = -4,\ c = 2$

x

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

The fraction bar extends under *–b.*

Use parentheses and substitute carefully to avoid errors.

Example 5

Using the Quadratic Formula

(Real Solutions)

5(a) Solve: $x^2 - 4x = -2$

Solution (cont'd):

$$x = \frac{4 \pm \sqrt{16 - 8}}{2}$$

Simplify.

$$x = \frac{4 \pm 2\sqrt{2}}{2}$$

$\sqrt{16-8} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$

Factor first, then divide.

$$x = \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$

Factor 2 out of the numerator.

$$x = 2 \pm \sqrt{2}$$

Lowest terms.

The solution set is $\{\mathbf{2 \pm \sqrt{2}}\}$.

Example 6
Using the Quadratic Formula
(Nonreal Complex Solutions)

6(a) Solve: $2x^2 = x - 4$

Solution:

$$2x^2 - x + 4 = 0$$

Standard form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula.
$a = 2$, $b = -1$, $c = 4$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(4)}}{2(2)}$$

The fraction bar extends under *–b.*

Use parentheses and substitute carefully to avoid errors.

Example 6

Using the Quadratic Formula (Nonreal Complex Solutions)

6(a) Solve: $2x^2 = x - 4$

Solution (cont'd):

$$x = \frac{1 \pm \sqrt{1 - 32}}{4}$$ Simplify.

$$x = \frac{1 \pm \sqrt{-31}}{4}$$ $\sqrt{-1} = i$

$$x = \frac{1 \pm i\sqrt{31}}{4}$$

The solution set is $\left\{\frac{1}{4} \pm \frac{\sqrt{31}}{4} i\right\}$.

CUBIC EQUATIONS

(Third Degree Equations)

CUBIC EQUATIONS

A ***cubic equation*** is a ***third-degree equation***, because the greatest degree of the terms is 3.

$x^3 = -8 \qquad 8x^3 + 10x^2 - 4x - 5 = 0 \qquad 3x^3 = 3x^2 - 9$

TERRAMETRA

Example 7
Solving a Cubic Equation

7(a) Solve: $x^3 + 8 - 0$ using factoring and the quadratic formula.

Solution:

$$(x + 2)(x^2 - 2x + 4) = 0$$ Factor as a sum of cubes.

$$x + 2 = 0 \text{ or } x^2 - 2x + 4 = 0$$ Zero-factor property.

$$x = -2 \text{ or } x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$ Quadratic formula. $a = 1,\ b = -2,\ c = 4$

Remember to include in the final solution set.

$$x = \frac{2 \pm \sqrt{-12}}{2}$$ Simplify.

TERRAMETRA

Example 7
Solving a Cubic Equation

7(a) Solve: $x^3 + 8 = 0$

Solution (cont'd):

$$x = \frac{2 \pm \sqrt{-12}}{2}$$

$$x = \frac{2 \pm 2i\sqrt{3}}{2}$$ Simplify the radical.

$$x = \frac{2\left(1 \pm i\sqrt{3}\right)}{2}$$ Factor 2 out of the numerator.

$$x = 1 \pm i\sqrt{3}$$ Lowest terms.

The solution set is $\{-2, 1 \pm i\sqrt{3}\}$.

Example 8

Solving for a Quadratic Variable

8(a) Solve the equation for the specified variable. Use $\pm$ when taking square roots.

Solve: $A = \frac{\pi d^2}{4}$, for d

Goal: Isolate *d*, the specified variable.

Solution:

$$A = \frac{\pi d^2}{4}$$

$$4A = \pi d^2$$ Multiply by 4 *(both sides)*.

$$\frac{4A}{\pi} = d^2$$ Divide by π *(both sides)*.

$$d = \pm\sqrt{\frac{4A}{\pi}}$$ Square root property.

See the Note following this example.

Example 8
Solving for a Quadratic Variable

8(a) Solve the equation for the specified variable. Use ± when taking square roots.

Solve: $A = \frac{\pi d^2}{4}$, for d

Solution (cont'd):

$$d = \pm\frac{\sqrt{4A}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}}$$ Multiply by $\frac{\sqrt{\pi}}{\sqrt{\pi}}$

$$d = \frac{\pm\sqrt{4A\pi}}{\pi}$$ Multiply numerators. Multiply denominators.

$$d = \frac{\pm 2\sqrt{A\pi}}{\pi}$$ Simplify the radical.

Example 8

Solving for a Quadratic Variable

8(b) Solve the equation for the specified variable.
Use ± when taking square roots.

Solve: $rt^2 - st = k \quad (r \neq 0)$, for t

Solution: Because $rt^2 - st = k$ has terms with t^2 and t, use the quadratic formula.

$$rt^2 - st - k = 0$$

Write in standard form.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula.
$a = r, b = -s, = -k$

TERRAMETRA

Example 8

Solving for a Quadratic Variable

8(b) Solve the equation for the specified variable. Use ± when taking square roots.

Solve: $rt^2 - st = k \quad (r \neq 0),$ for t

Solution (cont'd):

$$t = \frac{-(-s) \pm \sqrt{(-s)^2 - 4(r)(-k)}}{2(r)}$$

Quadratic formula. $a = r,\ b = -s,\ = -k$

$$t = \frac{s \pm \sqrt{s^2 + 4rk}}{2r}$$

Simplify.

Example 8
Solving for a Quadratic Variable

Note

In **Example 8** …
we took both positive and negative square roots.

However, if the variable represents time or length in an application, we consider only the *positive* square root.

DISCRIMINANT

DISCRIMINANT

The quantity under the radical in the quadratic formula, $b^2 - 4ac$, is called the ***discriminant***.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DISCRIMINANT

DISCRIMINANT	NUMBER OF SOLUTIONS	TYPE OF SOLUTIONS
Positive, Perfect Square	Two	Rational
Positive, Not a Perfect Square	Two	Irrational
Zero	One (Double Solution)	Rational
Negative	Two	Nonreal Complex

DISCRIMINANT

Caution

The restriction on a, b, and c is important.

For example $x^2 - \sqrt{5}x - 1 = 0$ has discriminant $b^2 - 4ac = 5 + 4 = 9$, which would indicate two rational solutions *if the coefficients were integers*.

By the quadratic formula, the two solutions …

$\frac{\sqrt{5} \pm 3}{2}$ are *irrational* numbers.

Example 9

Using the Discriminant

9(a) Evaluate the discriminant for the equation. Then use it to determine the number of distinct solutions, and tell whether they are *rational, irrational*, or *nonreal complex* numbers.

$$5x^2 + 2x - 4 = 0$$

Solution:

For $5x^2 + 2x - 4 = 0$, use $a = 5$, $b = 2$, and $c = -4$.

$$b^2 - 4ac = 2^2 - 4(5)(-4) = 84$$

The discriminant 84 is positive and not a perfect square …
There are two distinct irrational solutions.

Example 9
Using the Discriminant

9(b) Evaluate the discriminant for the equation. Then use it to determine the number of distinct solutions, and tell whether they are *rational, irrational*, or *nonreal complex* numbers.

$$x^2 - 10x = -25$$

Solution:

For $x^2 - 10x + 25 = 0$, use $a = 1$, $b = -10$, and $c = 25$.

$$b^2 - 4ac = 10^2 - 4(1)(25) = 0$$

There is one distinct rational solution, a double solution.

TERRAMETRA

Example 9
Using the Discriminant

9(c) Evaluate the discriminant for the equation. Then use it to determine the number of distinct solutions, and tell whether they are *rational, irrational*, or *nonreal complex* numbers.

$$2x^2 - x + 1 = 0$$

Solution:

For $2x^2 - x + 1 = 0$, use $a = 2$, $b = -1$, and $c = 1$.

$$b^2 - 4ac = (-1)^2 - 4(2)(1) = -7$$

There are two distinct nonreal complex solutions.
(They are complex conjugates.)