COMPLEX NUMBERS

Terrametra Resources

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1.3
COMPLEX NUMBERS

- Basic Concepts
- Operations on Complex Numbers

BASIC CONCEPTS

There is no real number solution of the equation

$$x^2 = -1$$

since no real number, when squared, gives –1.

To extend the real number system to include solutions of equations of this type, the number $\boldsymbol{i}$ is defined to have the following property …

$i = \sqrt{-1}$ therefore $i^2 = -1$

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BASIC CONCEPTS

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, then any number of the form $\boldsymbol{a+bi}$ is a ***complex number***.

In the complex number $\boldsymbol{a+bi}$,
$\boldsymbol{a}$ is the ***real part*** and $\boldsymbol{b}$ is the ***imaginary part***.

BASIC CONCEPTS

Two complex numbers $\boldsymbol{a + bi}$ and $\boldsymbol{c + di}$ are equal, provided that their real parts are equal and their imaginary parts are equal, …

$\boldsymbol{a + bi = c + di}$ if and only if $\boldsymbol{a = c}$ and $\boldsymbol{b = d}$

BASIC CONCEPTS

For complex number $\boldsymbol{a} + \boldsymbol{bi}$,
if $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} + \boldsymbol{bi} = \boldsymbol{a}$, …

Thus, the set of real numbers is a subset
of the set of complex numbers.

BASIC CONCEPTS

If $a = 0$ and $b \neq 0$, the complex number is said to be a ***pure imaginary number***.

A pure imaginary number, or a number like $7 + 2i$ with $a \neq 0$ and $b \neq 0$, is a ***nonreal complex number***.

A complex number written in the form $a + bi$ (or $a + ib$) is in ***standard form***.

BASIC CONCEPTS

Complex Numbers $a + bi$, for a and b Real

Real numbers
$a + bi, \ b = 0$

Rational numbers
$\frac{4}{9}, \ -\frac{5}{8}, \ \frac{11}{7}$

Integers
$-11, \ -6, \ -3, \ -2, \ -1$

Whole numbers
0

Natural numbers
$1, 2, 3, 4, 5, 37, 40$

Irrational numbers
$\sqrt{2}$
$\sqrt{15}$
$-\sqrt{8}$
π
$\frac{\pi}{4}$

Nonreal complex numbers
$a + bi, \ b \neq 0$
$7 + 2i, \ 5 - i\sqrt{3}, \ -\frac{1}{2} + \frac{3}{2}i$

Pure imaginary numbers
$a + bi, \ a = 0$ and $b \neq 0$
$3i, \ -i, \ -\frac{2}{3}i, \ i\sqrt{5}$

THE EXPRESSION $\sqrt{-a}$

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If $\boldsymbol{a} > \boldsymbol{0}$, then $\sqrt{-\boldsymbol{a}} = \boldsymbol{i}\sqrt{\boldsymbol{a}}$.

Example 1
Writing $\sqrt{-a}$ as $i\sqrt{a}$

1(a) Write as the product of a real number and $\boldsymbol{i}$, using the definition of $\sqrt{-\boldsymbol{a}}$.

$$\sqrt{-16}$$

Solution:

$$\sqrt{-16} = i\sqrt{16} = 4i$$

Example 1

Writing $\sqrt{-a}$ as $i\sqrt{a}$

1(b) Write as the product of a real number and i, using the definition of $\sqrt{-a}$.

$$\sqrt{-70}$$

Solution:

$$\sqrt{-70} = i\sqrt{70} = i\sqrt{70}$$

Example 1

Writing $\sqrt{-a}$ as $i\sqrt{a}$

1(c) Write as the product of a real number and $\boldsymbol{i}$, using the definition of $\sqrt{-\boldsymbol{a}}$.

$$\sqrt{-48}$$

Solution:

$$\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}$$

OPERATIONS on COMPLEX NUMBERS

Products or quotients with negative radicands are simplified by first rewriting $\sqrt{-a}$ as $i\sqrt{a}$ for a positive number a, …

Then the properties of real numbers and the fact that $i^2 = -1$ are applied.

OPERATIONS on COMPLEX NUMBERS

Caution

When working with negative radicands,
use the definition $\sqrt{-a} = i\sqrt{a}$
before using any of the other rules for radicals.

In particular, the rule

$$\sqrt{b} \cdot \sqrt{d} = \sqrt{bd}$$

is valid only when b and d are not both negative.

$\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$ is correct,

while

$\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$ is incorrect.

Example 2
Finding Products and Quotients Involving $\sqrt{-a}$

2(a) Multiply or divide, as indicated. Simplify the answer.

$$\sqrt{-7} \cdot \sqrt{-7}$$

Solution:

First write all square roots in terms of *i*.

$$\sqrt{-7} \cdot \sqrt{-7} = i\sqrt{7} \cdot i\sqrt{7}$$

$$= i^2\left(\sqrt{7}\right)^2 = i^2 \cdot 7 = -7 \qquad i^2 = -1$$

Example 2

Finding Products and Quotients Involving $\sqrt{-a}$

2(b) Multiply or divide, as indicated. Simplify the answer.

$$\sqrt{-6} \cdot \sqrt{-10}$$

Solution:

$$\sqrt{-6} \cdot \sqrt{-10} = i\sqrt{6} \cdot i\sqrt{10} = -1\sqrt{60}$$

$$= -1\sqrt{4 \cdot 15} = -1 \cdot 2\sqrt{15} = -2\sqrt{15}$$

Example 2

Finding Products and Quotients Involving $\sqrt{-a}$

2(c) Multiply or divide, as indicated. Simplify the answer.

$$\frac{\sqrt{-20}}{\sqrt{-2}}$$

Solution:

$$\frac{\sqrt{-20}}{\sqrt{-2}} = \frac{i\sqrt{20}}{i\sqrt{2}} = \sqrt{\frac{20}{2}} = \sqrt{10}$$

Quotient rule for radicals.

Example 2

Finding Products and Quotients Involving $\sqrt{-a}$

2(d) Multiply or divide, as indicated. Simplify the answer.

$$\frac{\sqrt{-48}}{\sqrt{24}}$$

Solution:

$$\frac{\sqrt{-48}}{\sqrt{24}} = \frac{i\sqrt{48}}{\sqrt{24}} = i\sqrt{\frac{48}{24}} = i\sqrt{2}$$

Quotient rule for radicals.

Example 3

Simplifying a Quotient Involving $\sqrt{-a}$

3(a) Write $\dfrac{-8+\sqrt{-128}}{4}$ in standard form $a + bi$.

Solution:

$$= \frac{-8 + \sqrt{-64 \cdot 2}}{4}$$

$$= \frac{-8 + 8i\sqrt{2}}{4}$$

$\sqrt{-64} = 8i$

Be sure to factor before simplifying

$$= \frac{4\left(-2 + 2i\sqrt{2}\right)}{4}$$

Factor.

$$= -2 + 2i\sqrt{2}$$

Lowest terms.

ADDITION and SUBTRACTION of COMPLEX NUMBERS

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For complex numbers $\boldsymbol{a+bi}$ and $\boldsymbol{c+di}$,

$$\boldsymbol{(a+bi)+(c+di)=(a+c)+(b+d)i}$$

and

$$\boldsymbol{(a+bi)-(c+di)=(a-c)+(b-d)i}$$

Example 4

Addition and Subtraction of Complex Numbers

4(a) Find the sum or difference.
Write answer in standard form.

$$(3 - 4i) + (-2 + 6i)$$

Solution:

Add real parts

Add imaginary parts

$$(3 - 4i) + (-2 + 6i) = [3 + (-2)] + [-4 + 6]i$$

Commutative, associative, distributive properties.

$$= 1 + 2i$$

Standard form.

Addition and Subtraction of Complex Numbers

4(b) Find the sum or difference.
Write answer in standard form.

$$(-4 + 3i) - (6 - 7i)$$

Solution:

$$(-4 + 3i) - (6 - 7i) = (-4 - 6) + [3 - (-7)]i$$

$$= -10 + 10i$$ Standard form.

MULTIPLICATION of COMPLEX NUMBERS

MULTIPLICATION of COMPLEX NUMBERS

For complex numbers $\boldsymbol{a} + \boldsymbol{bi}$ and $\boldsymbol{c} + \boldsymbol{di}$,

$$(\boldsymbol{a} + \boldsymbol{bi})(\boldsymbol{c} + \boldsymbol{di}) = (\boldsymbol{ac} - \boldsymbol{bd}) + (\boldsymbol{ad} + \boldsymbol{bc})\boldsymbol{i}$$

MULTIPLICATION of COMPLEX NUMBERS

The product of two complex numbers is found by multiplying as though the numbers were binomials and using the fact that $i^2 = -1$, as follows:

$$(a+bi)(c+di) = ac + adi + bic + bidi$$

FOIL

$$= ac + adi + bci + bdi^2$$

Distributive property.

$i^2 = -1$

$$= ac + (ad+bc)i + bd(-1)$$

$$= (ac - bd) + (ad + bc)i$$

Example 5

Multiplication of Complex Numbers

5(a) Find the product.
Write answer in standard form.

$$(2-3i)(3+4i)$$

Solution:

$$(2-3i)(3+4i) = 2(3) + 2(4i) - 3i(3) - 3i(4i)$$ FOIL

$$= 6 + 8i - 9i - 12i^2$$ Multiply.

$$= 6 - i - 12(-1)$$ Combine like terms. $i^2 = -1$

$$= 18 - i$$ Standard form.

Example 5
Multiplication of Complex Numbers

5(b) Find the product.
Write answer in standard form.

$$(4 + 3i)^2$$

Remember to add twice the product of the two terms.

Solution:

$$(4 + 3i)^2 = 4^2 + 2(4)(3i) + (3i)^2$$ Square of a binomial.

$$= 16 + 24i + 9i^2$$ Multiply.

$$= 16 + 24i + 9(-1)$$ $i^2 = -1$

$$= 7 + 24i$$ Standard form.

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Example 5

Multiplication of Complex Numbers

5(c) Find the product.
Write answer in standard form.

$$(6 + 5i)(6 - 5i)$$

Solution:

$$(6 + 5i)(6 - 5i) = 6^2 - (5i)^2$$ Product of the sum and difference of two terms.

$$= 36 - 25(-1)$$ $i^2 = -1$

$$= 36 + 25$$ Multiply.

$$= 61 \quad \text{or} \quad 61 + 0i$$ Standard form.

Example 5

Multiplication of Complex Numbers

Example 5(c) showed that …

$$(6 + 5i)(6 - 5i) = 61$$

The numbers $6 + 5i$ and $6 - 5i$ differ only in the sign of their imaginary parts and are called ***complex conjugates***.

The product of a complex number and its conjugate is always a real number.

This product is the sum of the squares of the real and imaginary parts.

PROPERTY of COMPLEX CONJUGATES

PROPERTY of COMPLEX CONJUGATES

For complex numbers $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$(\boldsymbol{a}+\boldsymbol{bi})(\boldsymbol{a}-\boldsymbol{bi})=\boldsymbol{a}^2+\boldsymbol{b}^2$$

Example 6

Multiplication of Complex Numbers

6(a) Find the quotient.
Write answer in standard form.

$$\frac{3+2i}{5-i}$$

Solution:

$$\frac{3+2i}{5-i} = \frac{(3+2i)(5+i)}{(5-i)(5+i)}$$

Multiply by the complex conjugate of the denominator in both the numerator and the denominator.

$$= \frac{15+3i+10i+2i^2}{25-i^2}$$

Multiply.

Example 6

Multiplication of Complex Numbers

Solution (cont'd):

$$\frac{15 + 3i + 10i + 2i^2}{25 - i^2} = \frac{13 + 13i}{26}$$

Combine like terms.
$i^2 = -1$

$$= \frac{13}{26} + \frac{13i}{26}$$

$$\frac{a + bi}{c} = \frac{a}{c} + \frac{bi}{c}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Write in lowest terms and standard form.

Check:

$$\left(\frac{1}{2} + \frac{1}{2}i\right)(5 - i) = 3 + 2i$$

Example 6

Multiplication of Complex Numbers

6(b) Find the quotient.
Write answer in standard form.

$$\frac{3}{i}$$

Solution:

$$\frac{3}{i} = \frac{3(-i)}{i(-i)} = \frac{-3i}{-i^2} = \frac{-3i}{1}$$

$-i$ is the conjugate of i
and
$-i^2 = -(i^2) = -1(-1) = 1$

$$= -3i \quad \text{or} \quad 0 - 3i$$

Standard form.

Powers of i

Powers of $\boldsymbol{i}$ can be simplified using the facts …

$$\boldsymbol{i^2 = -1} \text{ and } \boldsymbol{i^4 = \left(i^2\right)^2 = (-1)^2 = 1}$$

$i^1 = i$	$i^5 = i$	$i^9 = i$
$i^2 = -1$	$i^6 = 1$	$i^{10} = 1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$ …