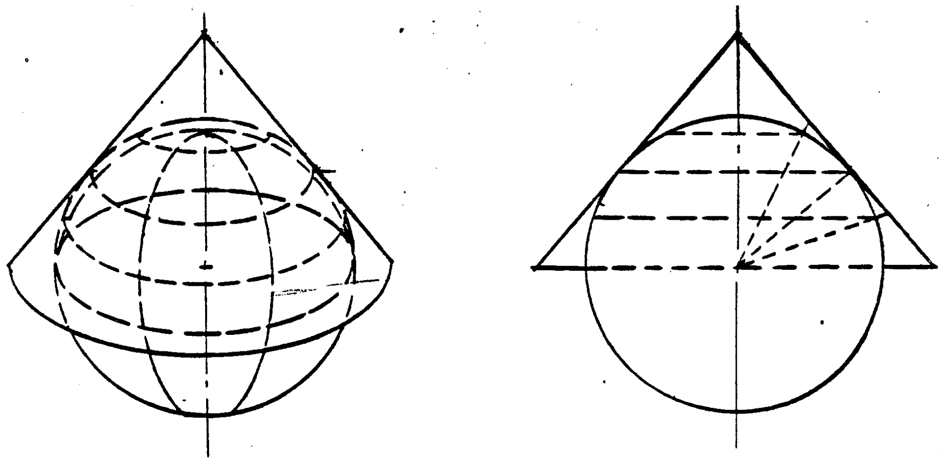
We will first consider a right circular cone that is tangent to a sphere on which circles of latitude and longitude have been drawn. This is shown in perspective in Figure 1A and in cross-section in Figure 1B. The cone could extend downward any distance, but we have cut it off even with the equator of the sphere to simplify the drawing that we will develop.



A B

Figure 1

Using our "projection light", we now project the latitude and longitude circles onto the cone and draw their "image lines". The circles of latitude will project onto the cone as circles, and the circles of longitude will project onto the cone as Straight lines. If we now cut the cone down the back side (along an element) and unroll it, it will look like Figure 2.

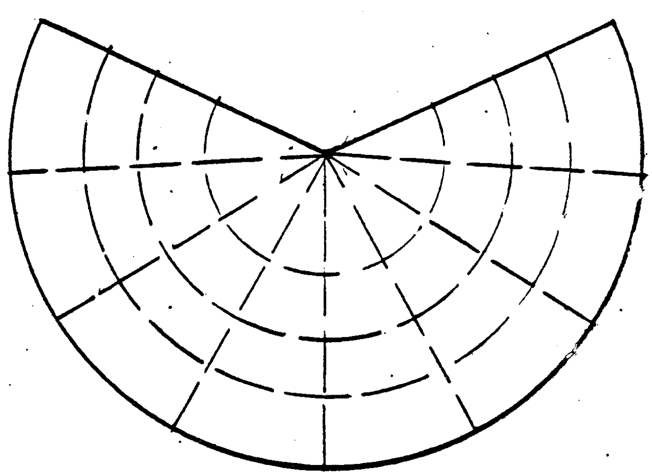


Figure 2

We now want to determine the relationship between the lines on the sphere and their images on the flat map. To help show this, let us present the cone by itself (without the image lines) in a perspective view, Figure 3A, and in its developed, or flat form, in Figure 3B.

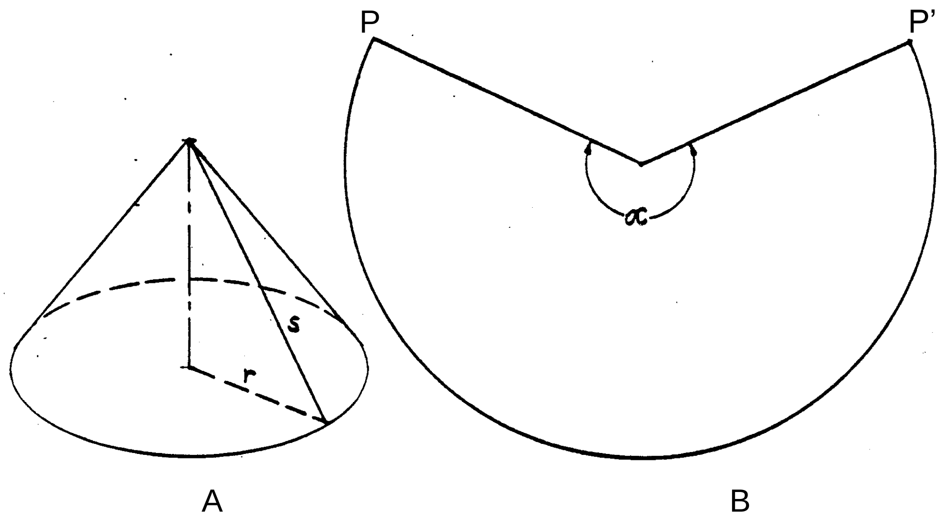


Figure 3

In Figure 3A, the circumference of the base of the cone equals 2πr. When the cone is cut and rolled out flat, this distance becomes the solid arc from P counterclockwise to P' in Figure 3B. The circumference of the whole circle in this figure equals 2πs. This distance (s) is the radius of the plane circle, but was formerly the <u>slant height</u> of the cone. Since in a plane circle an arc is directly proportional to its central angle:

$$\frac{2\pi r}{2\pi s} = \frac{\alpha}{360^\circ} \qquad \frac{r}{s} = \frac{\alpha}{360^\circ} \qquad \alpha = \frac{r}{s} \cdot 360^\circ$$

Hence we see that 360° of longitude on the sphere and cone is "compressed" in the map by the ratio of r/s. Any other angle representing the difference of longitude between two points would be compressed by the same ratio.

Referring to Figure 4, it can be seen that the ratio r/s is the sine of the vertex angle of the cone, that is, the angle at the central point between the axis of the cone and one of its elements. But this angle equals the latitude angle for the point of tangency between the sphere and the element, the angle known as phi (φ).

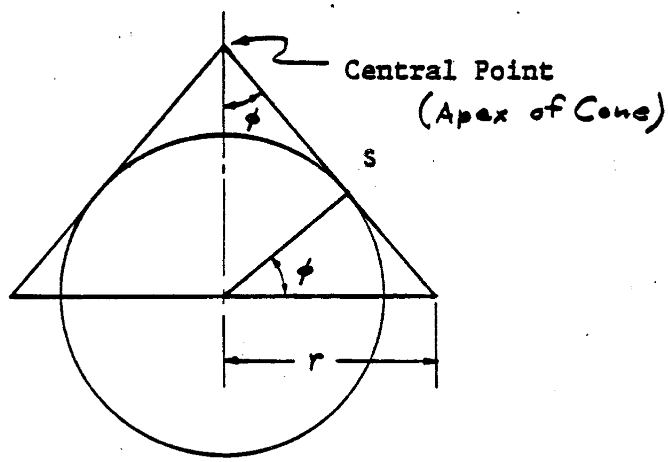


Figure 4

If we change the "steepness" of the side of the cone, we also change the latitude at which the cone will be tangent to the sphere, and, in the process we of course change α and the ratio r/s. In the case of a cone which is tangent to the sphere at 30° latitude, the developed surface will be a semi-circle, since $\alpha = 360° \cdot \sin(30°) = 180°$.

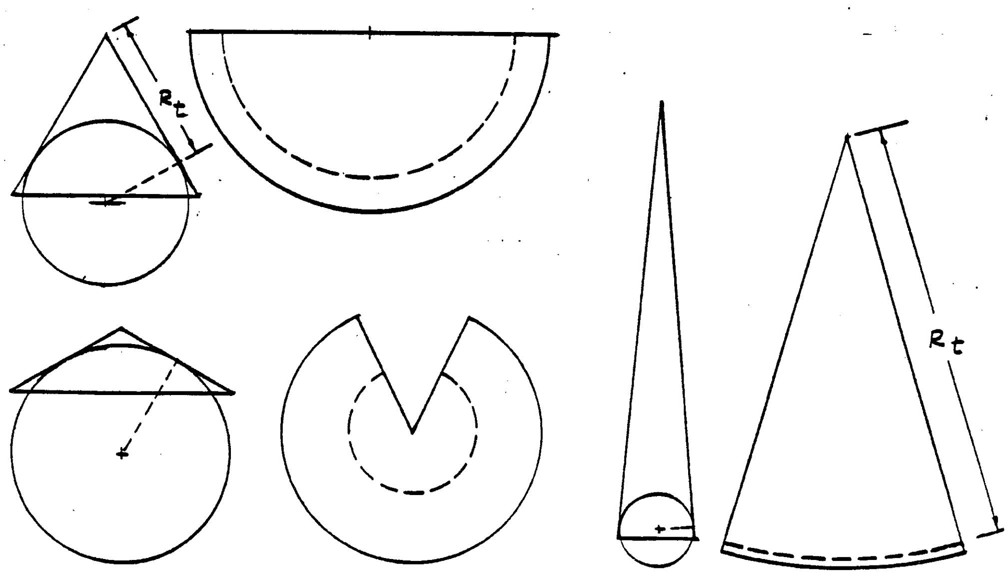


Figure 5

If the cone is tangent to the sphere between 0° and 30° latitude, the developed surface will be less than a semi-circle and R_t (the distance from the central point to the point of tangency) will become very large, whereas if the cone is tangent at a latitude greater than 30° but less than 90°, the developed surface will be more than a semi-circle but R_t will decrease in length. These various possibilities are shown in Figure 5 with the cones cut off slightly below the tangency line. Along the line of tangency (the standard parallel) the horizontal scale will be the same on the cone as on the sphere.

Similar relationships exist with the secant cone. However, since there are two lines of intersection of the cone and sphere, there will be two lines, or standard parallels, along which the horizontal distances are the same on the cone as on the sphere. This is shown in Figure 6.

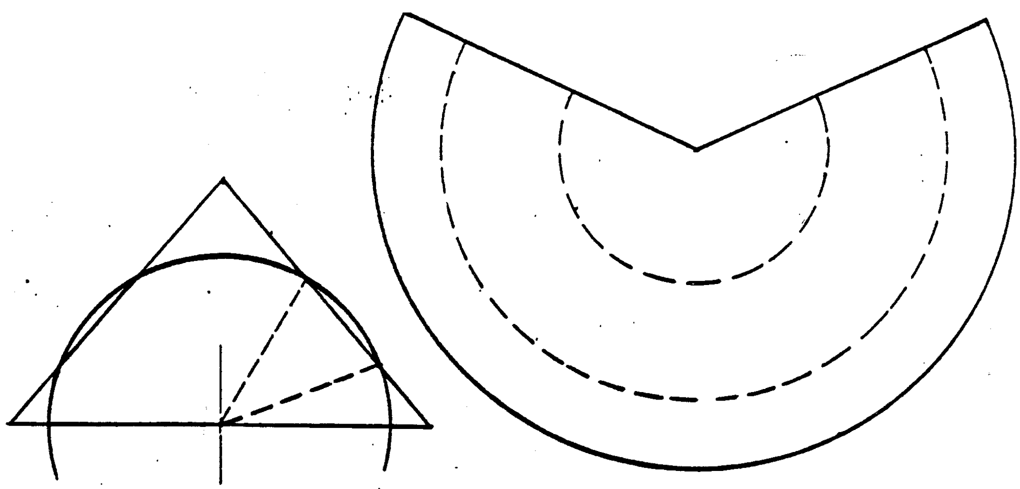


Figure 6

Points which have been projected from the sphere onto the cone can now be described in terms of plane polar coordinates, using the central point as the pole, and any element for the reference axis as shown in Figure 7. For convenience, the central meridian is normally used. This represents the element of the cone that appears vertical in the front view of the cone. The angle θ_1 and θ_2 are equal to the longitude angles ($\Delta\lambda$) between the central meridian and the points being plotted, multiplied by the factor r/s. (θ is sometimes called the mapping angle.) The distances ρ_1 and ρ_2 are the distances down the side of the cone from the central point. Hence, if we know the dimensions of the cone and sphere, and are given the latitude and longitude of any point on the sphere, we can plot its image point on the developed cone.

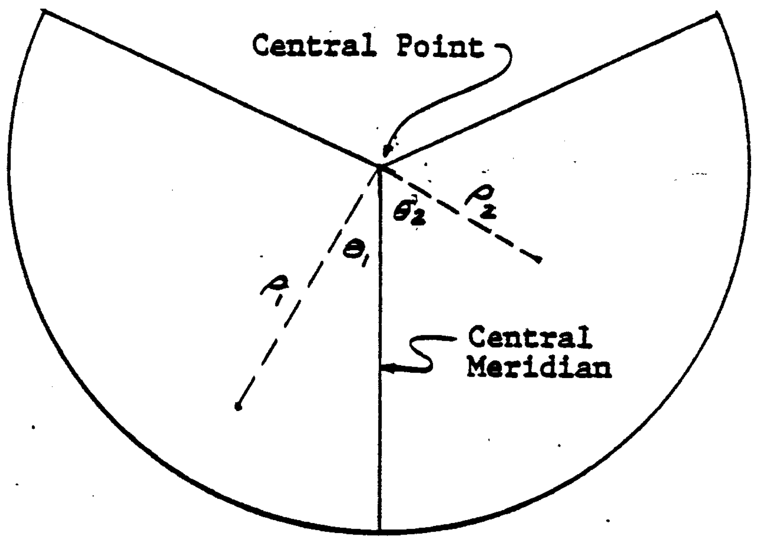


Figure 7

Once we have our points plotted, it is very simple to convert the positions to rectangular coordinates. Using the central meridian as the Y-axis and constructing a perpendicular to it at a convenient distance from the central point for the X-axis, we have the framework for a rectangular grid established as in Figure 8. As will be seen later the location of the Y-axis and X-axis must be accurately defined for the sake of uniformity. Using plane trigonometry we see, immediately that:

$$x_1 = -\rho_1 \cdot \sin(\theta_1) \qquad x_2 = \rho_2 \cdot \sin(\theta_2)$$
$$y_1 = R_b - \rho_1 \cdot \cos(\theta_1) \qquad y_2 = R_b - \rho_2 \cdot \cos(\theta_2)$$

x_1 is minus because θ_1 is left of the central meridian. This method for the development of plane rectangular coordinates works equally well for the tangent cone or secant cone. Sample problem 1 shows the tangent cone method and sample problem 2 shows the application to the secant cone.

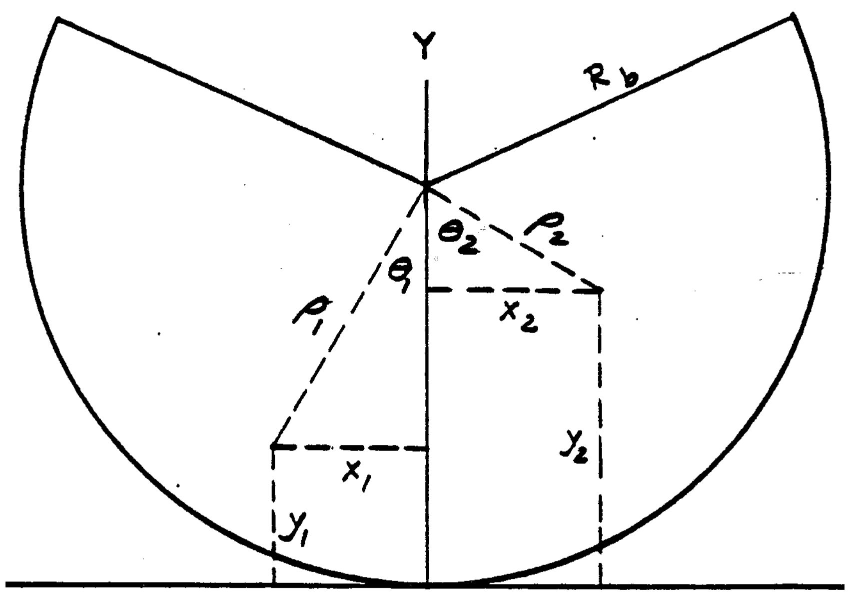


Figure 8

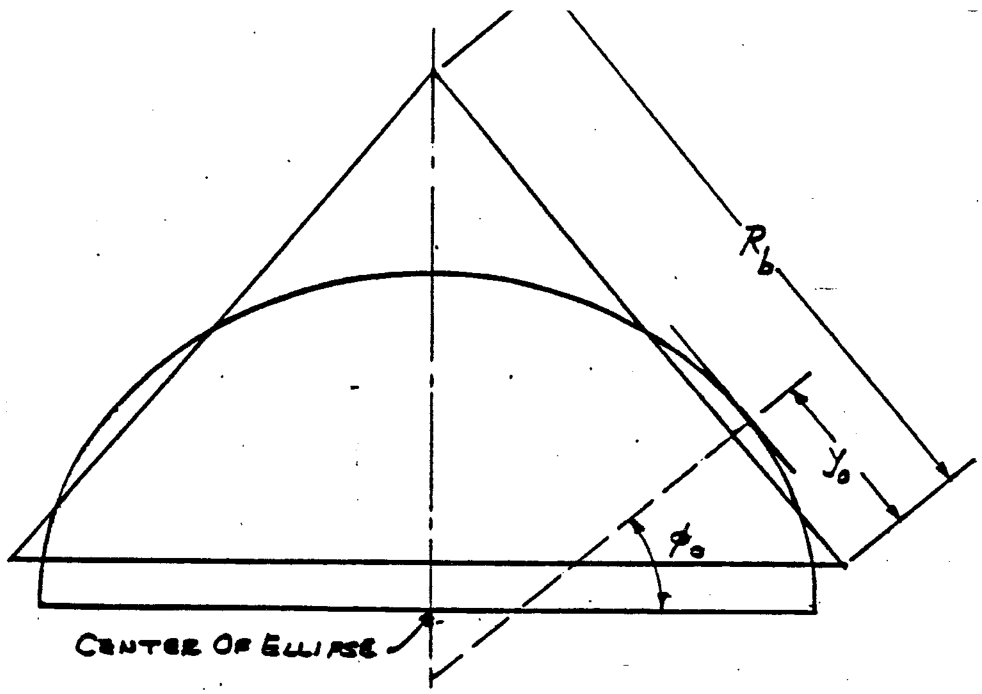


Figure 9

Up to this point we have confined our discussion to the sphere in order to simplify the trigonometry involved. In Figure 9 we see the secant cone method modified for the spheroid. This, of course, requires the use of an ellipse in the cross-section view, and as mentioned previously, complicates the geometric "projection" concept, since we cannot project all latitudes from the same center. In developing the tables for the state plane coordinate system, these variations were taken care of mathematically, but for all practical purposes the situation shown in Figure 9 represents graphically the Lambert type of projection.

When the conical surface is "developed" it looks very much like Figure 8. We now move the Y-axis to the left (west) a distance "C". This translation assures that all coordinates in the work area will be positive, both in Y and X. The final developed surface, or map, is shown in Figure 10.

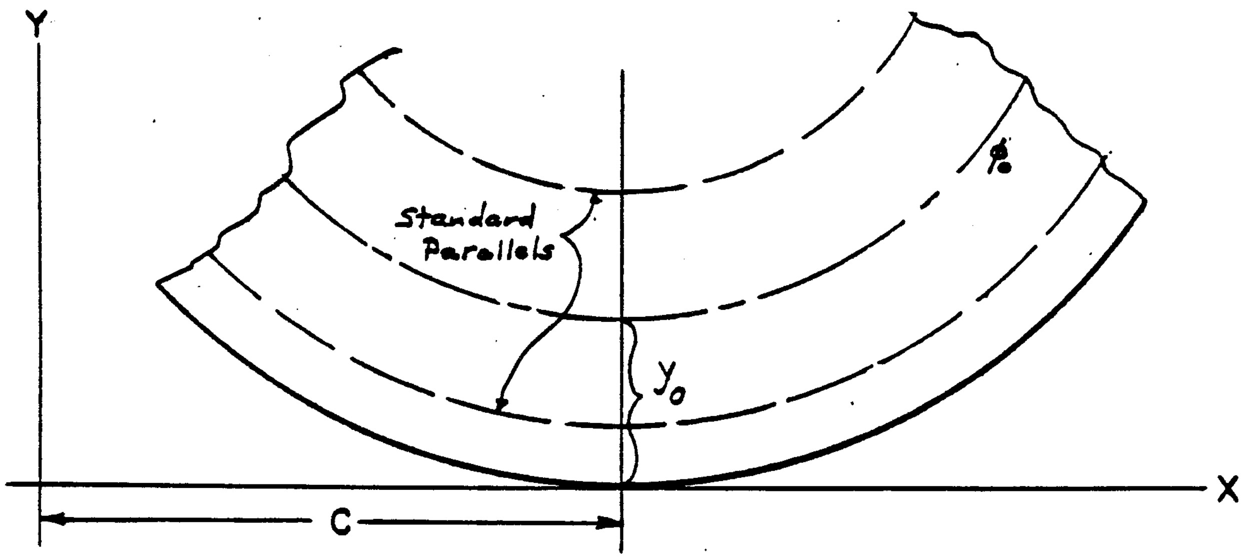


Figure 10

It should be noted that in Figure 9 the angle φ_o represents a latitude such that the normal to the tangent is also perpendicular to the element of the cone. When this latitude is projected onto the cone and the cone is developed, this latitude shows up in Figure 10 as the φ_o latitude line. This is close to, but not exactly half way between the standard parallels.