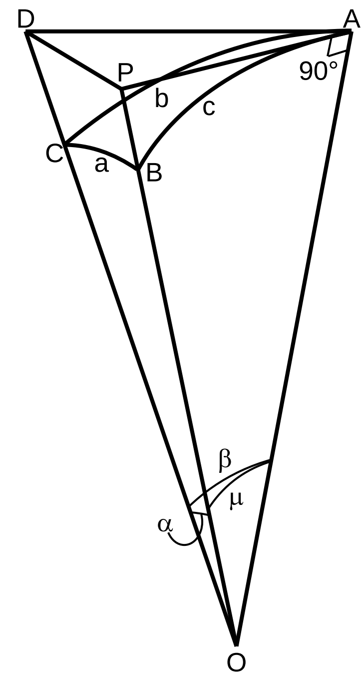
LAW of COSINES DERIVATION



Sides AD and AP are tangent at A such that…

DAP = spherical angle A
DAO = 90°
PAO = 90°

In plane triangle DPO…

Equation 1 $$DP^2 = PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos\alpha$$

In plane triangle ADP…

Equation 2 $$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In right triangle APO…

Equation 3 $$AO^2 = PO^2 - AP^2$$

In right triangle ADO…

Equation 4 $$AO^2 = DO^2 - AD^2$$

Equate the right sides of equations 1 and 2 and rearrange…

$$PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos\alpha = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 5 $$PO^2 + DO^2 - AD^2 - AP^2 = 2 \cdot PO \cdot DO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

Add equations 3 and 4 and rearrange…

$$AO^2 + AO^2 = PO^2 - AP^2 + DO^2 - AD^2$$

Equation 6 $$2 \cdot AO^2 = PO^2 + DO^2 - AD^2 - AP^2$$

Equate the left side of equation 6 to the right side of equation 5 and rearrange…

$$2 \cdot AO^2 = 2 \cdot PO \cdot DO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

$$AO^2 + AD \cdot AP \cdot \cos A = PO \cdot DO \cdot \cos\alpha$$

$$\cos\alpha = \frac{AO \cdot AO}{PO \cdot DO} + \frac{AP \cdot AD}{PO \cdot DO}\cos A \qquad = \cos\mu \cdot \cos\beta + \sin\mu \cdot \sin\beta \cdot \cos A$$

note… $\alpha = a$, $\beta = b$, $\mu = c$

Law of Cosines…

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$

alternate versions

LAW of SINES DERIVATION

Equation 7 $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$

Equation 8 $\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$

Equation 9 $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$

(Equations 7–9: by Law of Cosines)

Rearrange equations 7, 8 and 9…

Equation 10 $\cos a - \cos b \cdot \cos c = \cos A \cdot \sin b \cdot \sin c$ (from 7)

Equation 11 $\cos b - \cos a \cdot \cos c = \cos B \cdot \sin a \cdot \sin c$ (from 8)

Equation 12 $\cos c - \cos a \cdot \cos b = \cos C \cdot \sin a \cdot \sin b$ (from 9)

Using equation 10, square both sides…

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \cos^2 A \cdot \sin^2 b \cdot \sin^2 c$$

Apply the trig identity $\cos^2 \theta = 1 - \sin^2 \theta$ …

$$\left(1 - \sin^2 a\right) - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \left(1 - \sin^2 b\right) \cdot \left(1 - \sin^2 c\right) = \left(1 - \sin^2 A\right) \cdot \sin^2 b \cdot \sin^2 c$$

$$1 - \sin^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + 1 - \sin^2 c - \sin^2 b + \sin^2 b \cdot \sin^2 c = \sin^2 b \cdot \sin^2 c - \sin^2 A \cdot \sin^2 b \cdot \sin^2 c$$

Equation 13 $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 A \cdot \sin^2 b \cdot \sin^2 c$

Using equation 11 yields…

Equation 14 $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 B \cdot \sin^2 a \cdot \sin^2 c$

Using equation 12 yields…

Equation 15 $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 C \cdot \sin^2 a \cdot \sin^2 b$

Equate the right sides of equations 13, 14 and 15 and rearrange…

Law of Sines…

$$\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$$

LAW of COTANGENTS DERIVATION

Equation 16 $\sin b \cdot \sin A = \sin a \cdot \sin B$ … by Law of Sines

Equation 17 $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$

Equation 18 $\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$

Equation 19 $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$

(Equations 17–19) by Law of Cosines

Substitute for $\cos b$ from equation 18 into equation 17…

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity $\sin^2 \theta = 1 - \cos^2 \theta$ …

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

Rearrange and divide both sides with equation 16…

$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

Law of Cotangents…

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

Or …

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$

Spherical Excess…

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

where

$$s = \frac{a+b+c}{2}$$