

TRIGONOMETRIC
FUNCTIONS

$\sin(\theta) = \frac{O}{H}$
$\cos(\theta) = \frac{A}{H}$
$\tan(\theta) = \frac{O}{A}$
$\cot(\theta) = \frac{A}{O}$
$\sec(\theta) = \frac{H}{A}$
$\csc(\theta) = \frac{H}{O}$

RECIPROCAL
IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

RATIO
IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$	$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$

PYTHAGOREAN
IDENTITIES

$$\cos(\theta) = \frac{A}{H}$$

$$\sin(\theta) = \frac{O}{H}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

SYMMETRY
IDENTITIES

The graph of $\sin(\theta)$ is symmetric about the origin,
therefore...

$$\sin(-\theta) = -\sin(\theta)$$

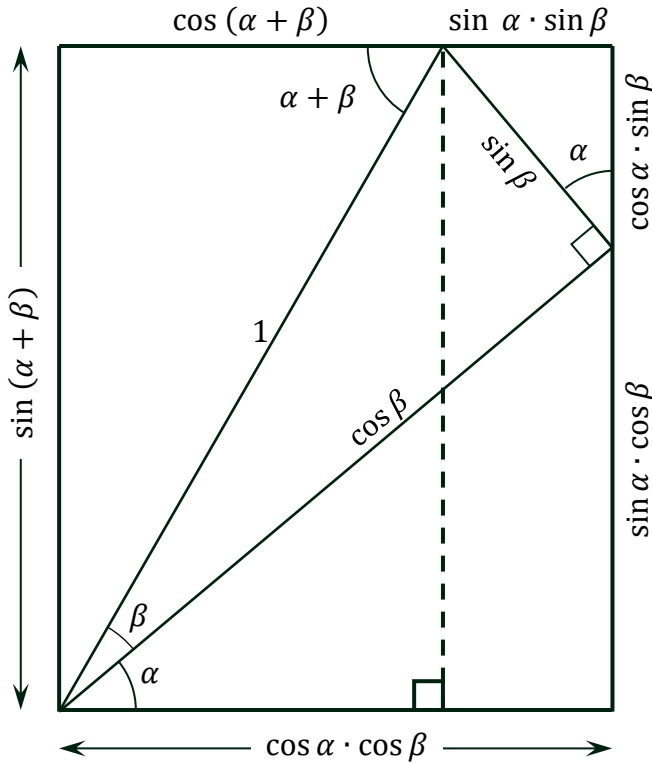
The graph of $\cos(\theta)$ is symmetric about the y-axis,
therefore...

$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin,
therefore...

$$\tan(-\theta) = -\tan(\theta)$$

SIN and COS of SUMS (and DIFFERENCES)



$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\ &= \sin \alpha \cdot \cos(-\beta) + \cos \alpha \cdot \sin(-\beta) \end{aligned}$$

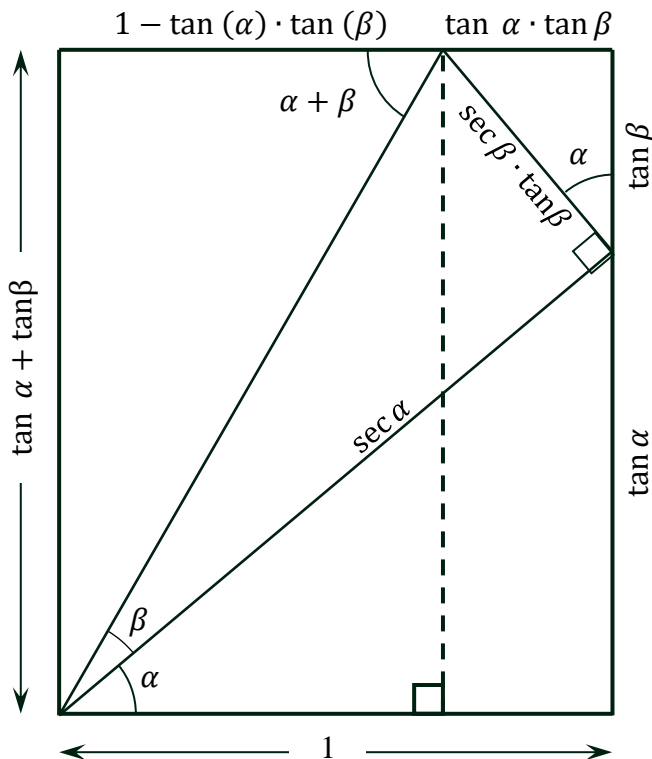
$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\begin{aligned} \cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\ &= \cos \alpha \cdot \cos(-\beta) - \sin \alpha \cdot \sin(-\beta) \end{aligned}$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

TAN of SUMS (and DIFFERENCES)



$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$\tan(-\beta) = -\tan \beta$$

$$\begin{aligned} \tan(\alpha - \beta) &= \tan(\alpha + (-\beta)) \\ &= \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \cdot \tan(-\beta)} \end{aligned}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$

DOUBLE ANGLE FORMULAE

$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(2\theta) = \sin(\theta + \theta) = \sin \theta \cdot \cos \theta + \cos \theta \cdot \sin \theta$$

$$\sin(2\theta) = 2 \cdot \sin \theta \cdot \cos \theta$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \cdot \sin^2 \theta$$

$$\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cdot \cos^2 \theta - 1$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \cdot \tan \theta}$$

$$\tan(2\theta) = \frac{2 \cdot \tan \theta}{1 - \tan^2 \theta} \cdot \left(\frac{\cos^2 \theta}{\cos^2 \theta} \right) =$$

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2 \cdot \sin \theta \cdot \cos \theta}{\cos^2 \theta - \sin^2 \theta}$$

HALF ANGLE FORMULAE

$$\cos(2\theta) = 1 - 2 \cdot \sin^2\theta$$

$$\cos(2\theta) = 2 \cdot \cos^2\theta - 1$$

$$2 \cdot \sin^2\theta = 1 - \cos(2\theta)$$

$$2 \cdot \cos^2\theta = 1 + \cos(2\theta)$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin \theta = \pm \sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\cos \theta = \pm \sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\frac{\theta}{2} = \theta$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos(\theta)}{2}}$$

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{\pm \sqrt{\frac{1 - \cos(\theta)}{2}}}{\pm \sqrt{\frac{1 + \cos(\theta)}{2}}}$$

$$\tan \frac{\theta}{2} = \frac{\sqrt{1 - \cos(\theta)}}{\sqrt{1 + \cos(\theta)}} \cdot \frac{\sqrt{1 + \cos(\theta)}}{\sqrt{1 + \cos(\theta)}} = \frac{\sqrt{1 - \cos^2(\theta)}}{1 + \cos(\theta)} = \frac{\sin(\theta)}{1 + \cos(\theta)}$$

$$\tan \frac{\theta}{2} = \frac{\sqrt{1 - \cos(\theta)}}{\sqrt{1 + \cos(\theta)}} \cdot \frac{\sqrt{1 - \cos(\theta)}}{\sqrt{1 - \cos(\theta)}} = \frac{1 - \cos(\theta)}{\sqrt{1 - \cos^2(\theta)}} = \frac{1 - \cos(\theta)}{\sin(\theta)}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

LAW of SINES DERIVATION

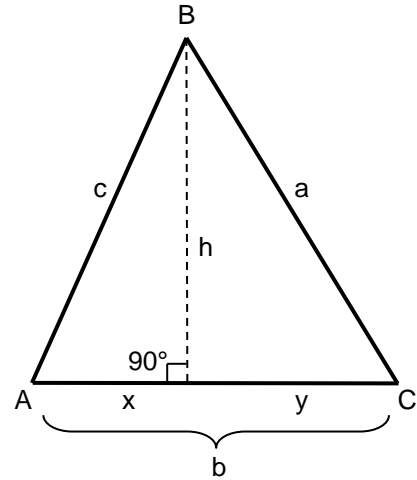
Equation 1 $h = c \cdot \sin A$

Equation 2 $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange...

Equation 3 $\frac{a}{\sin A} = \frac{c}{\sin C}$

Note: $\frac{a}{\sin A} = \frac{b}{\sin B}$ and $\frac{b}{\sin B} = \frac{c}{\sin C}$ are derived similarly.



Law of Sines...

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

LAW of COSINES DERIVATION

Equation 4 $h^2 = a^2 - y^2 = a^2 - (b-x)^2$

Equation 5 $h^2 = c^2 - x^2$

Equate the right sides of equations 4 and 5 and rearrange...

$$a^2 - (b-x)^2 = c^2 - x^2$$

Equation 6 $a^2 = b^2 + c^2 - 2bx$

Equation 7 $x = c \cdot \cos A$

Substitute for x from equation 7 into equation 6...

Equation 8 $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Note: equations for b^2 and c^2 are derived similarly.

Law of Cosines...

alternate versions

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

or

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

or

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$

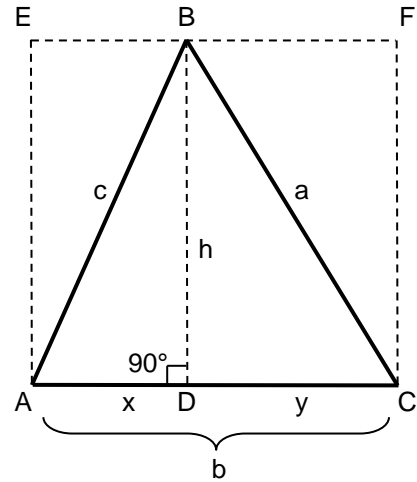
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

or

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

$$\text{Area ABD} = \text{Area ABE} = \frac{x \cdot h}{2}$$

$$\text{Area CBD} = \text{Area CBF} = \frac{y \cdot h}{2}$$



Therefore...

Area of Triangle ABC...

$$\text{Area} = \frac{b \cdot h}{2}$$

$$h = a \cdot \sin C$$

Therefore...

Area of Triangle ABC...

$$\text{Area} = \frac{a \cdot b \cdot \sin C}{2}$$

$$b = \frac{a \cdot \sin B}{\sin A}$$

Therefore...

Area of Triangle ABC...

$$\text{Area} = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$

HERON'S FORMULA DERIVATION

Equation 1

$$\text{Area} = \frac{b \cdot h}{2}$$

From triangle ABD...

$$x^2 + h^2 = c^2$$

Equation 2

$$x^2 = c^2 - h^2$$

Equation 3

$$x = \sqrt{c^2 - h^2}$$

From triangle CBD...

$$(b - x)^2 + h^2 = a^2$$

$$(b - x)^2 = a^2 - h^2$$

Equation 4

$$b^2 - 2bx + x^2 = a^2 - h^2$$

Substitute x and x^2 from Equations 3 and 2 into Equation 4 ...

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides ...

$$(b^2 + c^2 - a^2)^2 = 4b^2(c^2 - h^2)$$

$$\frac{(b^2 + c^2 - a^2)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{(b^2 + c^2 - a^2)^2}{4b^2}$$

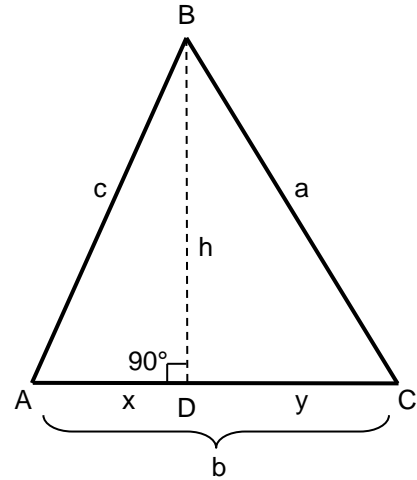
$$h^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2 - (b^2 + c^2 - a^2)^2}{4b^2}$$

$$h^2 = \frac{[2bc + (b^2 + c^2 - a^2)] \cdot [2bc - (b^2 + c^2 - a^2)]}{4b^2}$$

$$h^2 = \frac{[2bc + b^2 + c^2 - a^2] \cdot [2bc - b^2 - c^2 + a^2]}{4b^2}$$

$$h^2 = \frac{[(b^2 + 2bc + c^2) - a^2] \cdot [a^2 - (b^2 - 2bc + c^2)]}{4b^2}$$



$$h^2 = \frac{[(b+c)^2 - a^2] \cdot [a^2 - (b-c)^2]}{4b^2}$$

$$h^2 = \frac{[(b+c)+a] \cdot [(b+c)-a] \cdot [a+(b-c)] \cdot [a-(b-c)]}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}{4b^2}$$

Since $P = a + b + c \dots$

$$h^2 = \frac{P(P-2a)(P-2b)(P-2c)}{4b^2}$$

Equation 5

$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

Substitute h from Equation 5 into Equation 1 ...

$$Area = \frac{1}{2} b \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

$$Area = \frac{1}{4} \sqrt{P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\frac{1}{16} P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\left(\frac{P}{2}\right) \left(\frac{P-2a}{2}\right) \left(\frac{P-2b}{2}\right) \left(\frac{P-2c}{2}\right)}$$

$$Area = \sqrt{\frac{P}{2} \left(\frac{P}{2} - a\right) \left(\frac{P}{2} - b\right) \left(\frac{P}{2} - c\right)}$$

Thus...

Given the three sides of a triangle (a , b , and c) ...

The area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where the semi perimeter is:

$$s = P/2 = (a + b + c)/2$$

LAW of COSINES DERIVATION

Sides AD and AP are tangent at A such that...

$$\begin{aligned} \text{DAP} &= \text{spherical angle } A \\ \text{DAO} &= 90^\circ \\ \text{PAO} &= 90^\circ \end{aligned}$$

In plane triangle DPO...

Equation 1 $DP^2 = PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos \alpha$

In plane triangle ADP...

Equation 2 $DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$

In right triangle APO...

Equation 3 $AO^2 = PO^2 - AP^2$

In right triangle ADO...

Equation 4 $AO^2 = DO^2 - AD^2$

Equate the right sides of equations 1 and 2 and rearrange...

$$PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos \alpha = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 5 $PO^2 + DO^2 - AD^2 - AP^2 = 2 \cdot PO \cdot DO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A$

Add equations 3 and 4 and rearrange...

$$AO^2 + AO^2 = PO^2 - AP^2 + DO^2 - AD^2$$

Equation 6 $2 \cdot AO^2 = PO^2 + DO^2 - AD^2 - AP^2$

Equate the left side of equation 6 to the right side of equation 5 and rearrange...

$$\begin{aligned} 2 \cdot AO^2 &= 2 \cdot PO \cdot DO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A \\ AO^2 + AD \cdot AP \cdot \cos A &= PO \cdot DO \cdot \cos \alpha \end{aligned}$$

$$\cos \alpha = \frac{AO \cdot AO}{PO \cdot DO} + \frac{AP \cdot AD}{PO \cdot DO} \cos A = \cos \mu \cdot \cos \beta + \sin \mu \cdot \sin \beta \cdot \cos A$$

note... $\alpha = a, \beta = b, \mu = c$

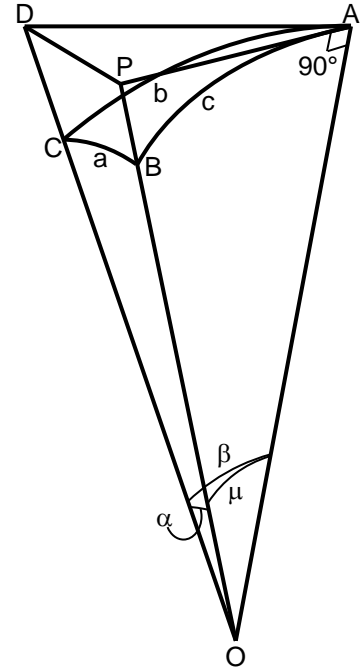
Law of Cosines...

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$

} alternate versions



LAW of SINES DERIVATION

<u>Equation 7</u>	$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$	}	by Law of Cosines
<u>Equation 8</u>	$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$		
<u>Equation 9</u>	$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$		

Rearrange equations 7, 8 and 9...

<u>Equation 10</u>	$\cos a - \cos b \cdot \cos c = \cos A \cdot \sin b \cdot \sin c$	(from 7)
<u>Equation 11</u>	$\cos b - \cos a \cdot \cos c = \cos B \cdot \sin a \cdot \sin c$	(from 8)
<u>Equation 12</u>	$\cos c - \cos a \cdot \cos b = \cos C \cdot \sin a \cdot \sin b$	(from 9)

Using equation 10, square both sides...

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \cos^2 A \cdot \sin^2 b \cdot \sin^2 c$$

Apply the trig identity $\cos^2 \theta = 1 - \sin^2 \theta \dots$

$$(1 - \sin^2 a) - 2 \cdot \cos a \cdot \cos b \cdot \cos c + (1 - \sin^2 b) \cdot (1 - \sin^2 c) = (1 - \sin^2 A) \cdot \sin^2 b \cdot \sin^2 c$$

$$1 - \sin^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + 1 - \sin^2 c - \sin^2 b + \sin^2 b \cdot \sin^2 c = \sin^2 b \cdot \sin^2 c - \sin^2 A \cdot \sin^2 b \cdot \sin^2 c$$

$$\text{Equation 13} \quad 2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 A \cdot \sin^2 b \cdot \sin^2 c$$

Using equation 11 yields...

$$\text{Equation 14} \quad 2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 B \cdot \sin^2 a \cdot \sin^2 c$$

Using equation 12 yields...

$$\text{Equation 15} \quad 2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 C \cdot \sin^2 a \cdot \sin^2 b$$

Equate the right sides of equations 13, 14 and 15 and rearrange...

Law of Sines...

$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$

LAW of COTANGENTS DERIVATION

Equation 16 $\sin b \cdot \sin A = \sin a \cdot \sin B$... by Law of Sines

Equation 17 $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$

Equation 18 $\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$

Equation 19 $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$

} by Law of Cosines

Substitute for $\cos b$ from equation 18 into equation 17...

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity $\sin^2 \theta = 1 - \cos^2 \theta$...

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

Rearrange and divide both sides with equation 16...

$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

Law of Cotangents...

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

Or ...

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$

Spherical Excess...

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

where

$$s = \frac{a+b+c}{2}$$