



# *SPHERICAL TRIGONOMETRY*

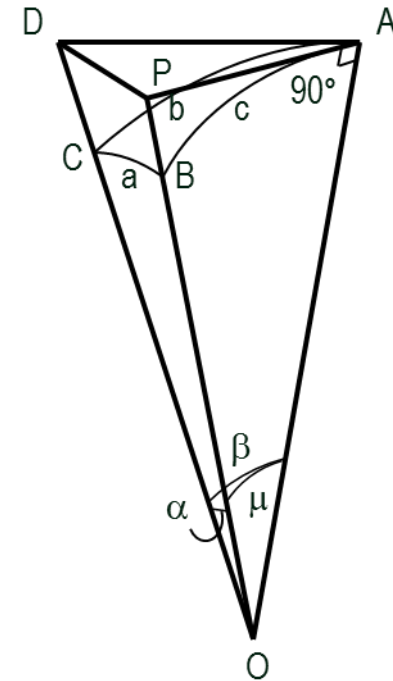
Terrametra Resources

Lynn Patten



# The Spherical Trig “*LAW*S”

- ▼ *Law of Cosines*
- ▼ *Law of Sines*
- ▼ *Law of Cotangents*
- ▼ *Spherical Excess*

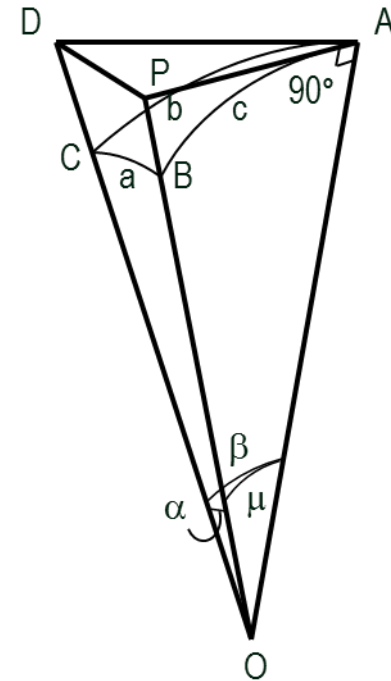




# Basic Assumptions

Sides AD and AP are tangent at A  
such that ...

- $DAP = \text{spherical angle } A$
- $DAO = 90^\circ$
- $PAO = 90^\circ$





# Plane Trig Formulae

In plane triangle DPO ...

✔ **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha$$

In plane triangle DPA ...

✔ **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In plane triangle ADO ...

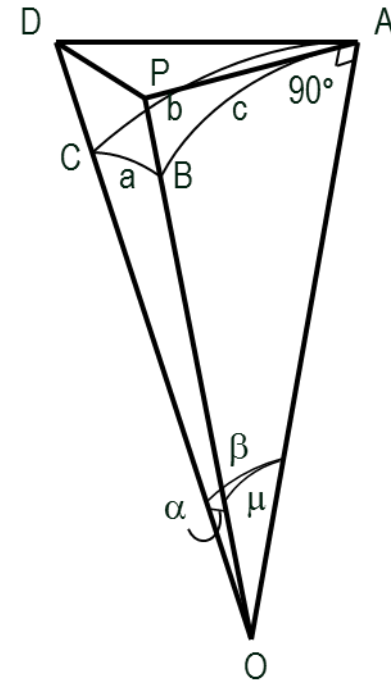
✔ **Equation 3**

$$DO^2 = AD^2 + AO^2$$

In plane triangle APO ...

✔ **Equation 4**

$$PO^2 = AP^2 + AO^2$$





✔ **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha$$

✔ **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

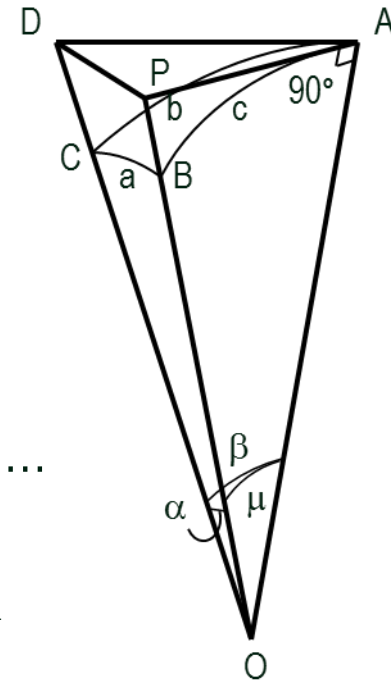
Equate the right sides of equations 1 and 2 and rearrange ...

$$DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha =$$

$$AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

✔ **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A$$





✔ **Equation 3**

$$DO^2 = AD^2 + AO^2$$

✔ **Equation 4**

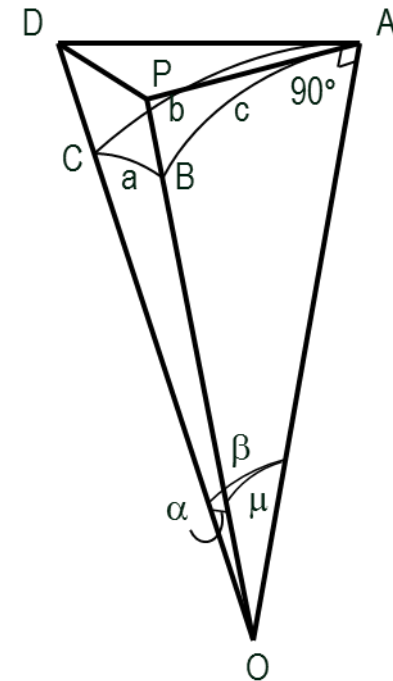
$$PO^2 = AP^2 + AO^2$$

Add equations 3 and 4 and rearrange ...

$$DO^2 + PO^2 = AD^2 + AP^2 + AO^2 + AO^2$$

✔ **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$





✔ **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

✔ **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$

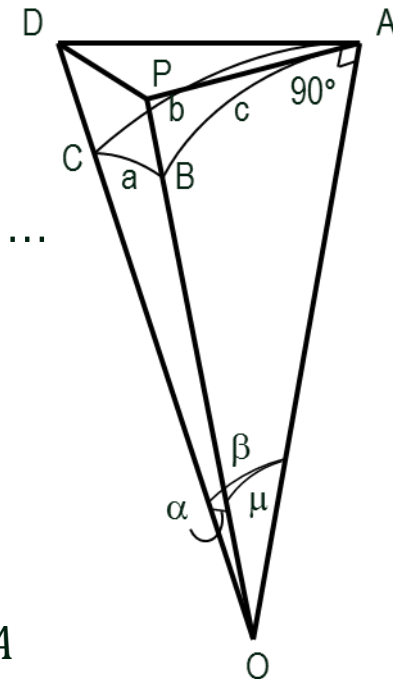
Equate the right sides of equations 5 and 6 and rearrange ...

$$2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A = 2 \cdot AO^2$$

$$DO \cdot PO \cdot \cos \alpha = AO \cdot AO + AD \cdot AP \cdot \cos A$$

$$\cos \alpha = \frac{AO \cdot AO}{DO \cdot PO} + \frac{AD \cdot AP}{DO \cdot PO} \cos A$$

$$\cos \alpha = \cos \beta \cdot \cos \mu + \sin \beta \cdot \sin \mu \cdot \cos A$$





# “LAW of COSINES”

$$\cos \alpha = \cos \beta \cdot \cos \mu + \sin \beta \cdot \sin \mu \cdot \cos A$$

Note ...  $\alpha = a$   $\beta = b$   $\mu = c$

## “LAW of COSINES”

### ✔ Equation 7

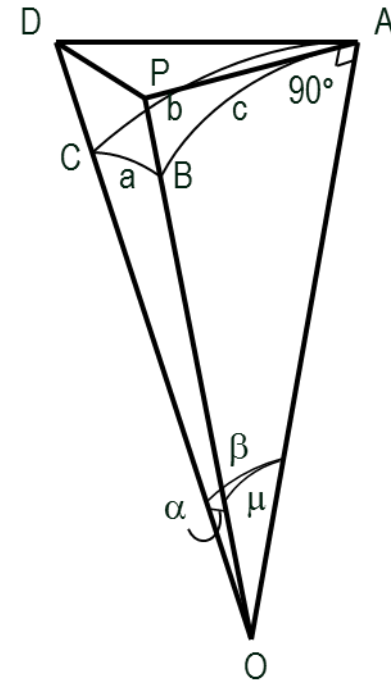
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

### ✔ Equation 8

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

### ✔ Equation 9

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$







Rearrange equations 7, 8 and 9 ...

✔ **Equation 10** (from equation 7)

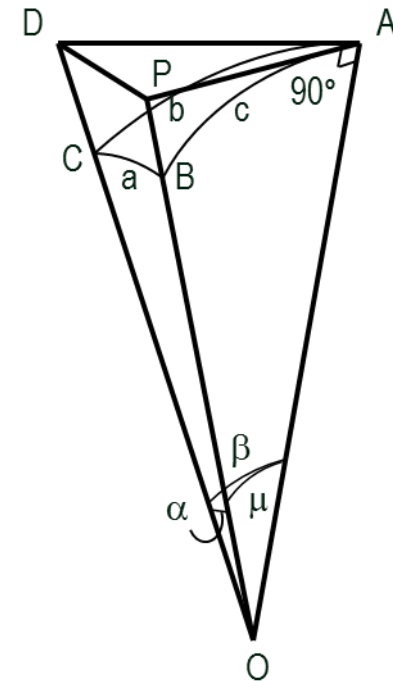
$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

✔ **Equation 11** (from equation 8)

$$\cos b - \cos a \cdot \cos c = \sin a \cdot \sin c \cdot \cos B$$

✔ **Equation 12** (from equation 9)

$$\cos c - \cos a \cdot \cos b = \sin a \cdot \sin b \cdot \cos C$$





✓ **Equation 10** (from equation 7)

$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

Square both sides of equation 10

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \sin^2 b \cdot \sin^2 c \cdot \cos^2 A$$

Apply the trig identity ...  $\cos^2 \theta = 1 - \sin^2 \theta$

$$(1 - \sin^2 a) - 2 \cos a \cos b \cos c + (1 - \sin^2 b)(1 - \sin^2 c) = \sin^2 b \sin^2 c (1 - \sin^2 A)$$

$$1 - \sin^2 a - 2 \cos a \cos b \cos c + 1 - \sin^2 b - \sin^2 c + \sin^2 b \sin^2 c =$$

$$\sin^2 b \sin^2 c - \sin^2 b \sin^2 c \sin^2 A$$

✓ **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$



# “LAW of SINES”

✓ **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

✓ **Equation 14** (from equation 11)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B$$

✓ **Equation 15** (from equation 12)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

Equate the right sides of equations 13, 14 and 15 and rearrange ...

$$-\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

$$\sin b \cdot \sin c \cdot \sin A = \sin a \cdot \sin c \cdot \sin B = \sin a \cdot \sin b \cdot \sin C$$

“LAW of SINES”

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



✔ **Equation 16** (Law of Sines)

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

✔ **Equation 17** (Law of Cosines)

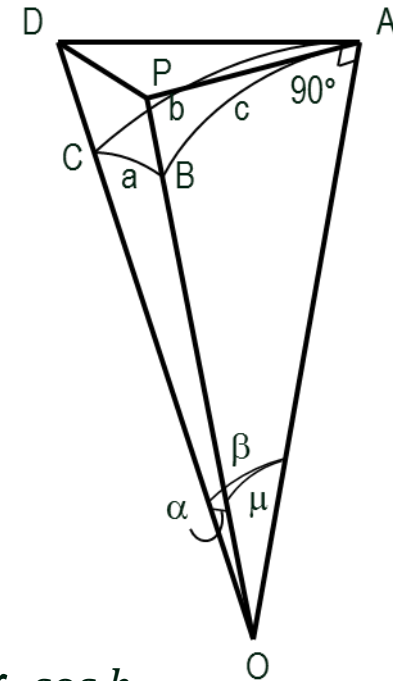
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

✔ **Equation 18** (Law of Cosines)

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

✔ **Equation 19** (Law of Cosines)

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$



Substitute the right side of equation 18 into equation 17 for  $\cos b$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$



$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity ...  $\sin^2 \theta = 1 - \cos^2 \theta$

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

✓ **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$



# “LAW of COTANGENTS”

## ✔ Equation 20

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

## ✔ Equation 16

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

Divide equation 16 into equation 20 and rearrange ...

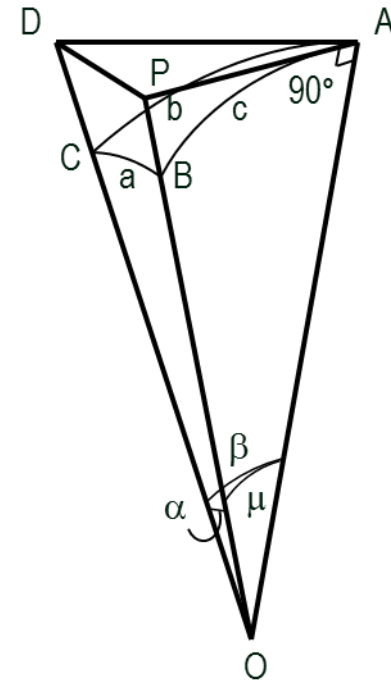
$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

## “LAW of COTANGENTS”

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

... or ...

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$





# Spherical Excess

## “SPHERICAL EXCESS”

$$E = A + B + C - 180^\circ$$

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

... where ...

$$s = \frac{a + b + c}{2}$$