TERRAMETRA

TRIGONOMETRY
“Plane” and Simple to “Spherical”

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Trigonometric Functions

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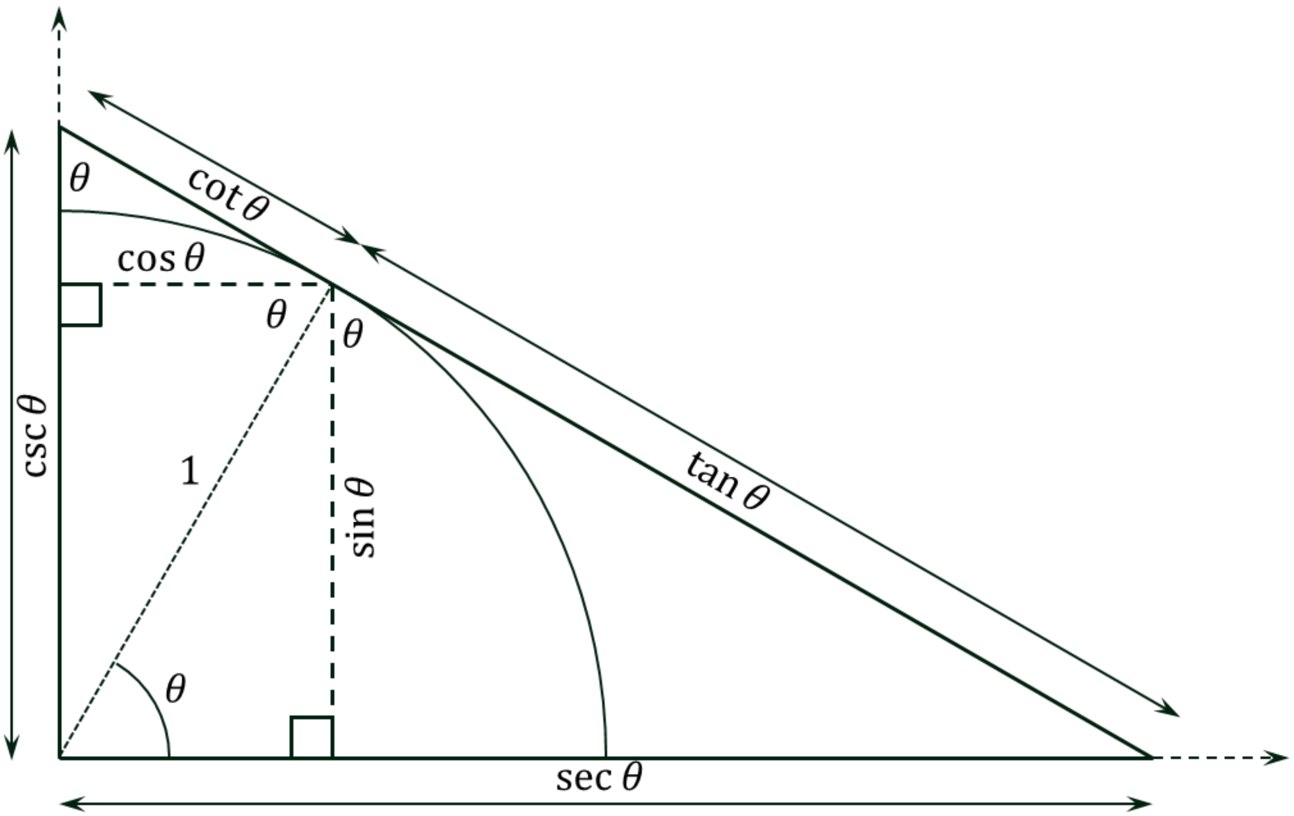
Trigonometric Functions



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

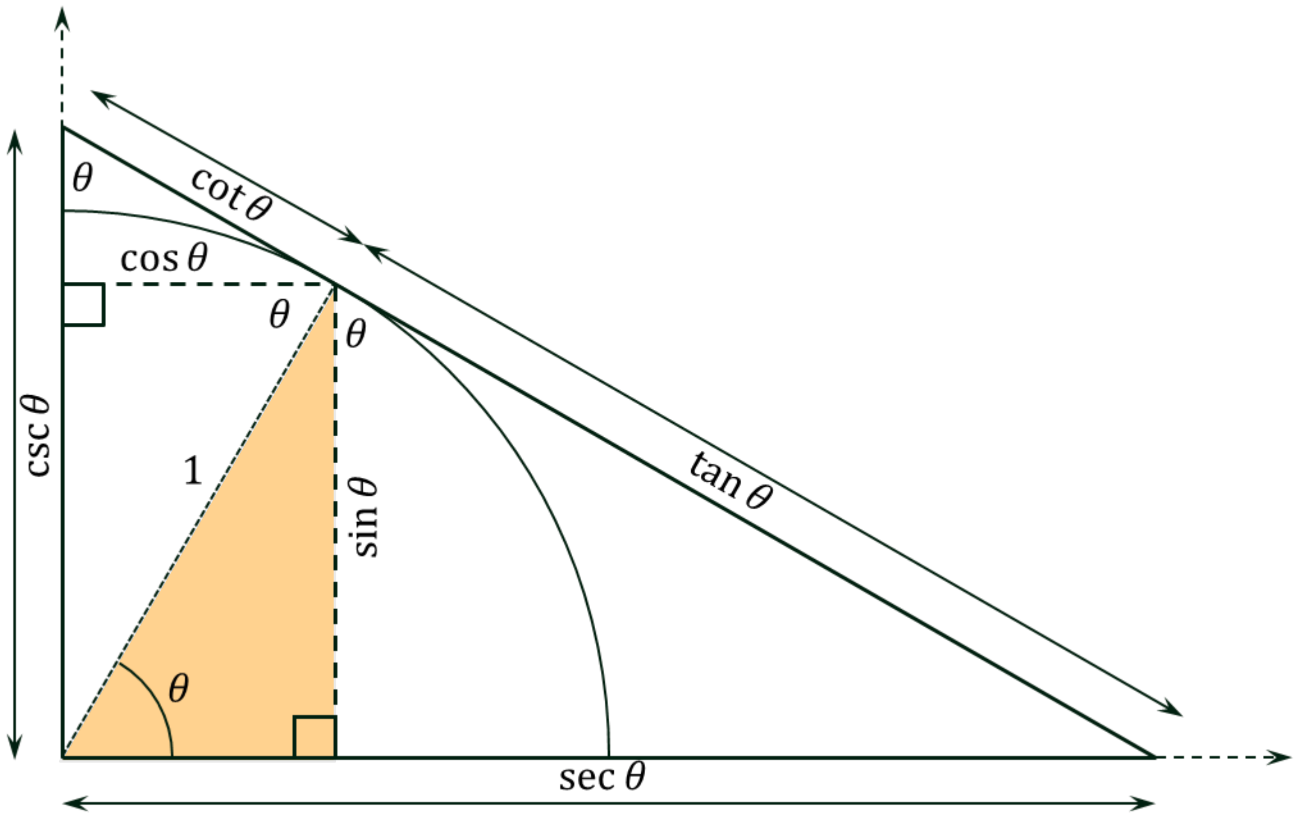
$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

SIN Function



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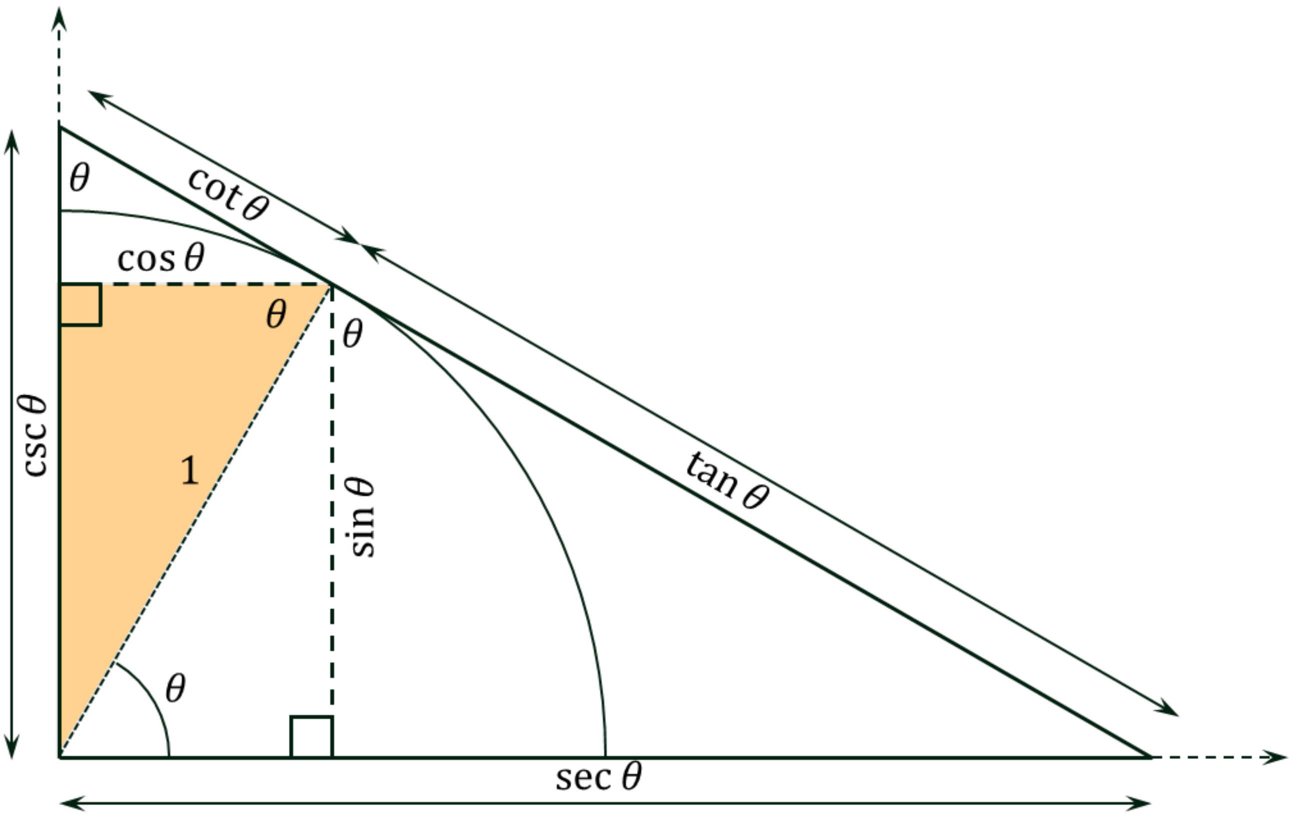
COS Function



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$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

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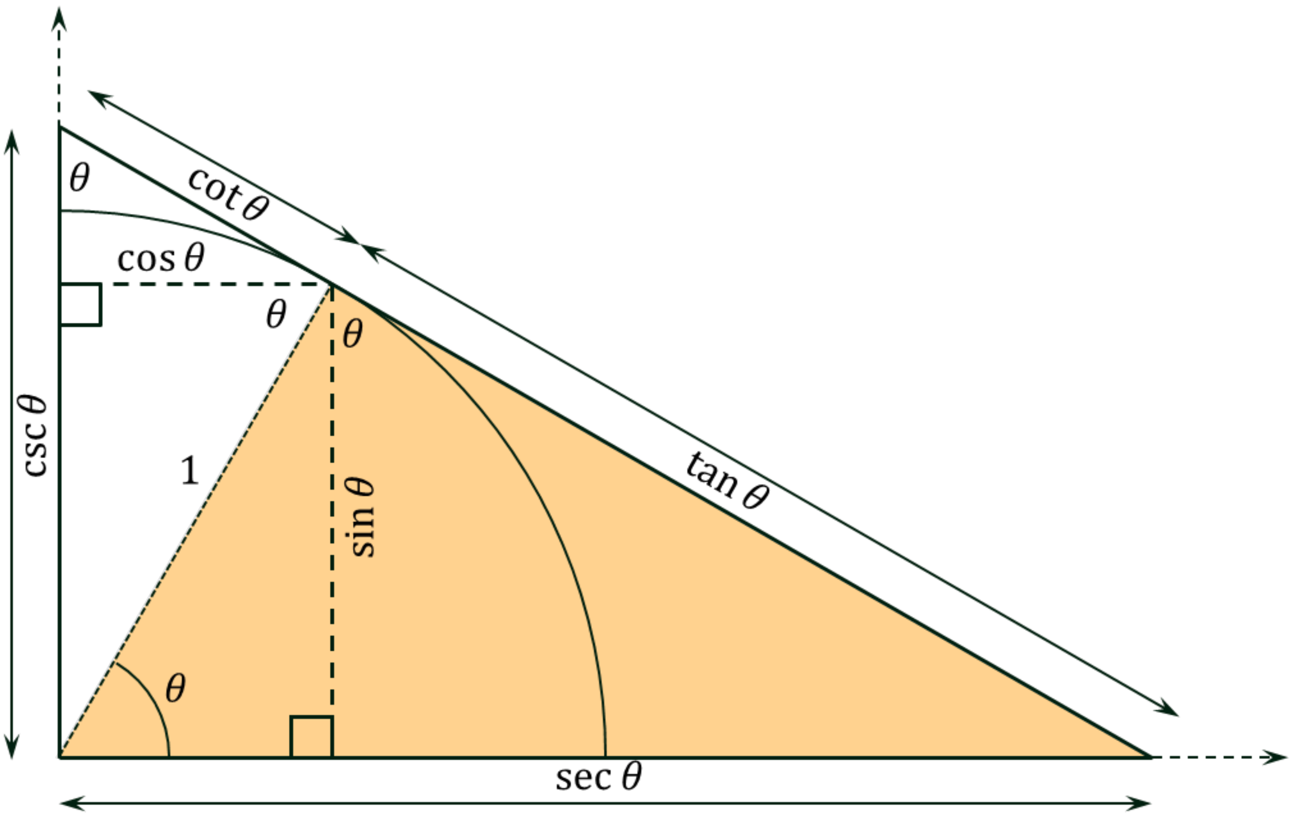
TAN Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

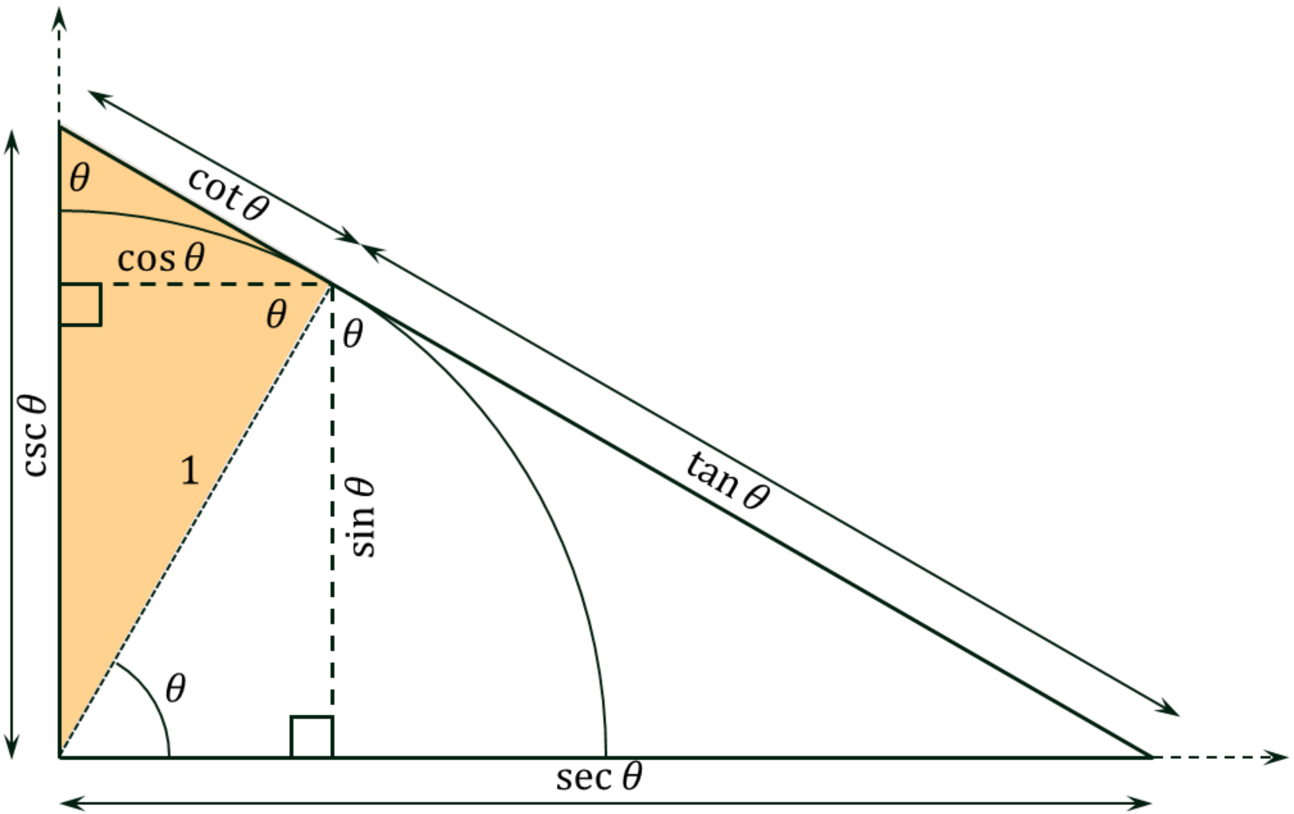
$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

COT Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

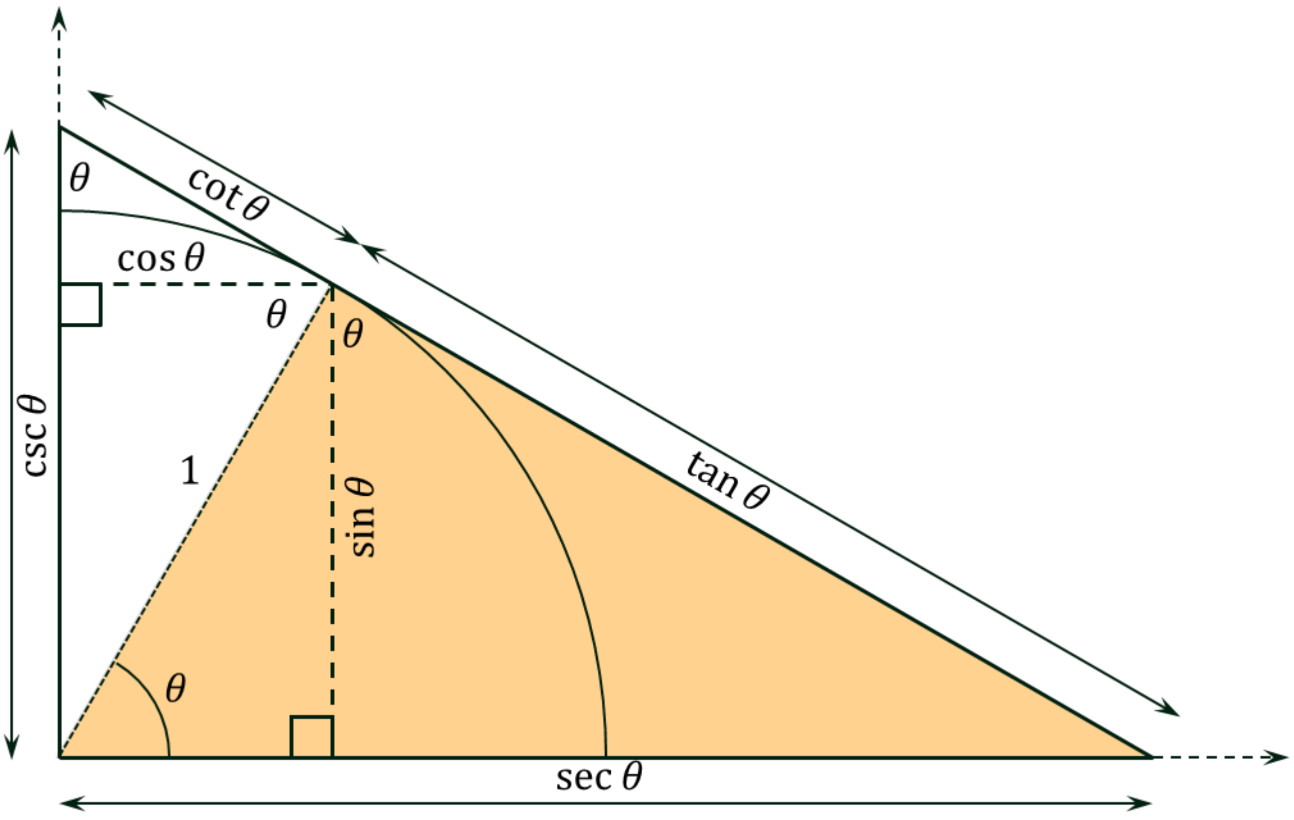
$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

SEC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

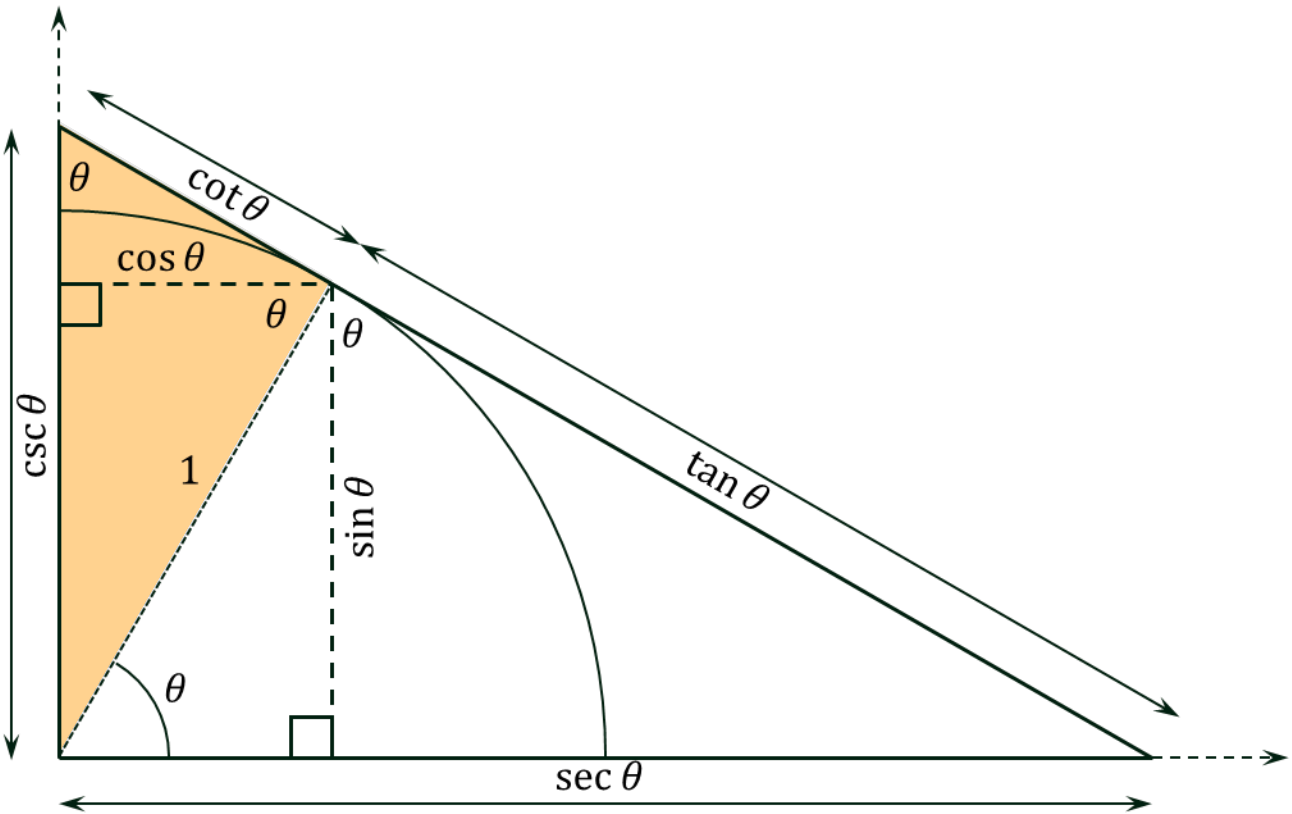
$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

CSC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

Trigonometric Identities

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Reciprocal Identities

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

Ratio Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$$

$$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$$

Pythagorean Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\cos(\theta) = \frac{A}{H}$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Symmetry Identities

The graph of $\sin(\theta)$ is symmetric about the origin, therefore…

$$\sin(-\theta) = -\sin(\theta)$$

The graph of $\cos(\theta)$ is symmetric about the y−axis, therefore…

$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin, therefore…

$$\tan(-\theta) = -\tan(\theta)$$

Derived Trigonometric Identities

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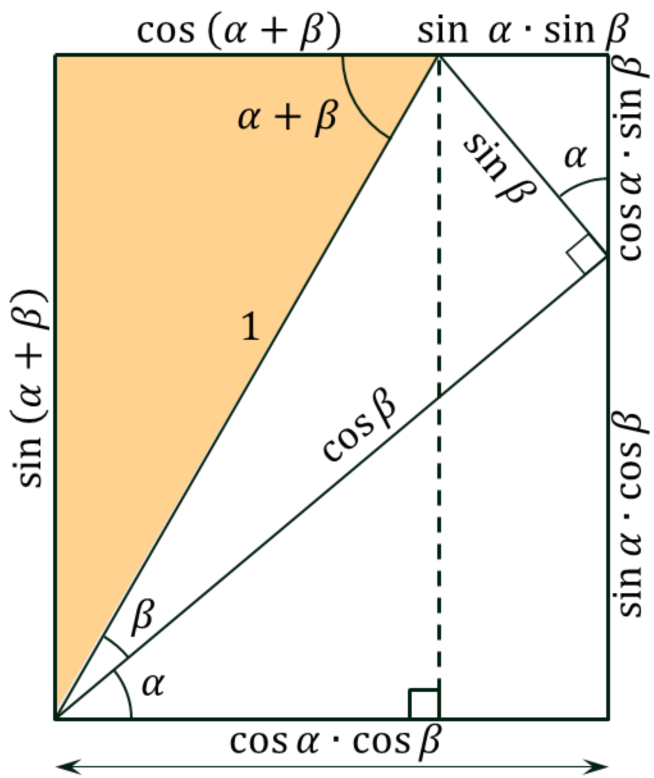
SIN of Sums (and Differences)



$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$\sin(-\beta) = -\sin\beta$$

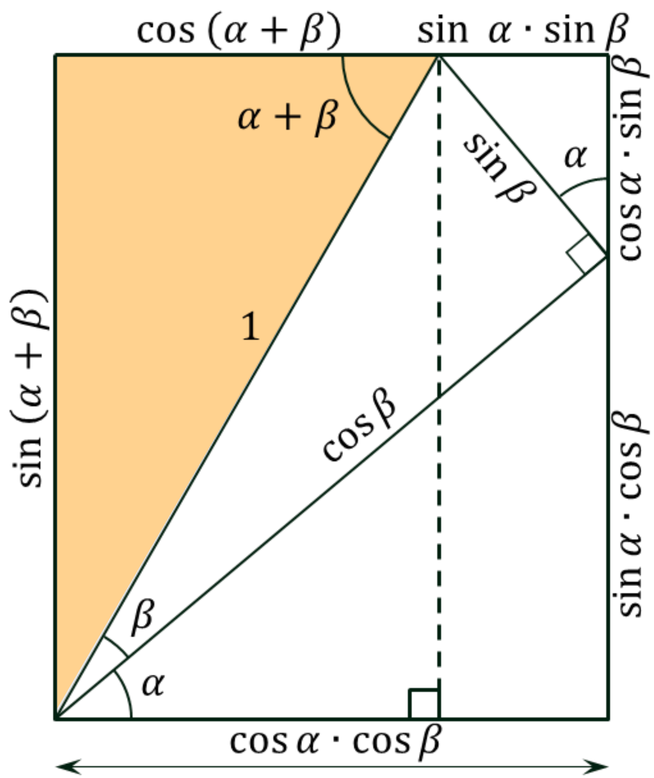
$$\cos(-\beta) = \cos\beta$$

$$\sin(\alpha-\beta) = \sin\big(\alpha+(-\beta)\big)$$

$$= \sin\alpha \cdot \cos(-\beta) + \cos\alpha \cdot \sin(-\beta)$$

$$\sin(\alpha-\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

COS of Sums (and Differences)



$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$$

$$\sin(-\beta) = -\sin\beta$$

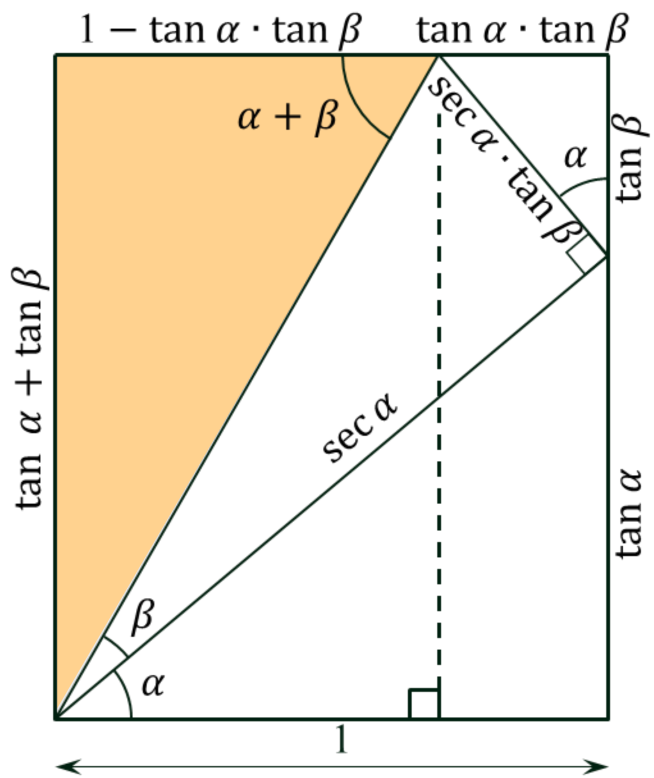
$$\cos(-\beta) = \cos\beta$$

$$\cos(\alpha-\beta) = \cos\big(\alpha+(-\beta)\big)$$

$$= \cos\alpha\cdot\cos(-\beta) - \sin\alpha\cdot\sin(-\beta)$$

$$\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$$

TAN of Sums (and Differences)



$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$\tan(-\beta) = -\tan\beta$$

$$\tan(\alpha - \beta) = \tan\big(\alpha + (-\beta)\big)$$

$$= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha \cdot \tan(-\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

Double Angle Formulae

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta$$

$$\boxed{\sin(2\theta) = 2 \cdot \sin\theta \cdot \cos\theta}$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta$$

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}$$

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta \quad \boxed{= 1 - 2 \cdot \sin^2\theta}$$

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) \quad \boxed{= 2 \cdot \cos^2\theta - 1}$$

Double Angle Formulae

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\tan(2\theta)=\tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\cdot\tan\theta}$$

$$\boxed{\tan(2\theta)=\frac{2\cdot\tan\theta}{1-\tan^2\theta}}\cdot\left(\frac{\cos^2\theta}{\cos^2\theta}\right)=$$

$$\tan(2\theta)=\frac{\sin(2\theta)}{\cos(2\theta)}\quad\boxed{=\frac{2\cdot\sin\theta\cdot\cos\theta}{\cos^2\theta-\sin^2\theta}}$$

Half Angle Formulae

$$\cos(2\theta) = 1 - 2 \cdot \sin^2\theta$$

$$2 \cdot \sin^2\theta = 1 - \cos(2\theta)$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\sin\frac{\emptyset}{2} = \pm\sqrt{\frac{1 - \cos(\emptyset)}{2}}$$

$$\frac{\emptyset}{2} = \theta$$

$$\cos(2\theta) = 2 \cdot \cos^2\theta - 1$$

$$2 \cdot \cos^2\theta = 1 + \cos(2\theta)$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\cos\theta = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\cos\frac{\emptyset}{2} = \pm\sqrt{\frac{1 + \cos(\emptyset)}{2}}$$

Half Angle Formulae

$$\tan\frac{\emptyset}{2} = \frac{\sin\frac{\emptyset}{2}}{\cos\frac{\emptyset}{2}} = \frac{\pm\sqrt{\frac{1-\cos(\emptyset)}{2}}}{\pm\sqrt{\frac{1+\cos(\emptyset)}{2}}}$$

$$\tan\frac{\emptyset}{2} = \pm\sqrt{\frac{1-\cos(\emptyset)}{1+\cos(\emptyset)}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}}\cdot\frac{\sqrt{1+\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}} = \frac{\sqrt{1-\cos^2(\emptyset)}}{1+\cos(\emptyset)} = \frac{\sin(\emptyset)}{1+\cos(\emptyset)}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}}\cdot\frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1-\cos(\emptyset)}} = \frac{1-\cos(\emptyset)}{\sqrt{1-\cos^2(\emptyset)}} = \frac{1-\cos(\emptyset)}{\sin(\emptyset)}$$

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Trigonometric Laws

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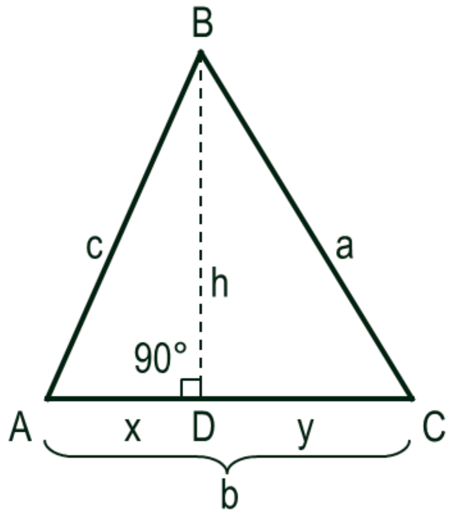
"LAW of SINES"

- **Equation 1** $h = c \cdot \sin A$
- **Equation 2** $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange …

- **Equation 3** $$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Similarly: $$\frac{a}{\sin A} = \frac{b}{\sin B}$$ and $$\frac{b}{\sin B} = \frac{c}{\sin C}$$



"LAW of SINES"

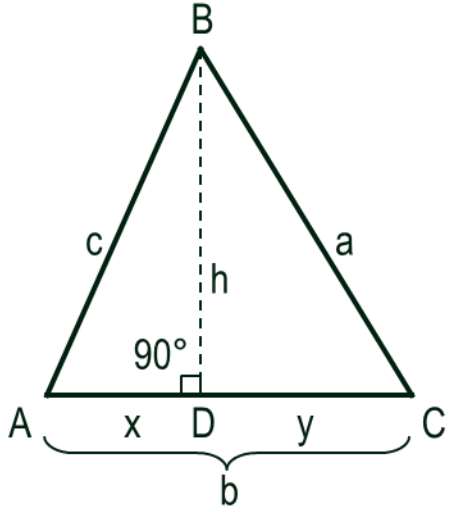
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

… or …

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

"LAW of COSINES"

- **Equation 1** $h^2 = c^2 - x^2$
- **Equation 2** $h^2 = a^2 - y^2 = a^2 - (b - x)^2$



Equate the right sides of equations 1 and 2 and rearrange …

$$a^2 - (b - x)^2 = c^2 - x^2$$

- **Equation 3** $a^2 = b^2 + c^2 - 2bx$
- **Equation 4** $x = c \cos A$

Substitute for x from equation 4 into equation 3…

"LAW of COSINES" $a^2 = b^2 + c^2 - 2bc \cos A$

Similarly:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

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Plane Triangle Areas

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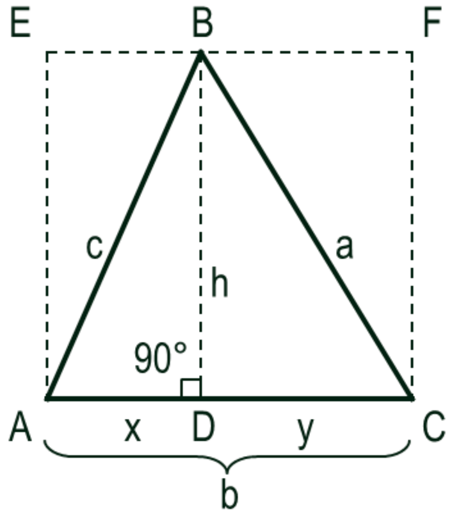
Plane Triangle Areas

$$Area\ ABD = Area\ ABE = \frac{x \cdot h}{2}$$

$$Area\ CBD = Area\ CBF = \frac{y \cdot h}{2}$$



- **Area of Triangle ABC…**

$$Area = \frac{b \cdot h}{2}$$

$$h = a\ \sin C$$

- **Area of Triangle ABC…**

$$Area = \frac{a \cdot b \cdot \sin C}{2}$$

$$b = \frac{a \cdot \sin B}{\sin A}$$

- **Area of Triangle ABC…**

$$Area = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$

Heron’s Formula

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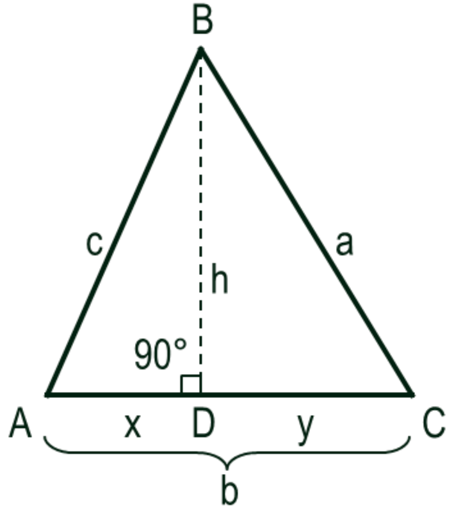
- Equation 1 $Area = \frac{b \cdot h}{2}$



From triangle ABD… $x^2 + h^2 = c^2$

- Equation 2 $x^2 = c^2 - h^2$
- Equation 3 $x = \sqrt{c^2 - h^2}$

From triangle CBD… $(b - x)^2 + h^2 = a^2$

$(b - x)^2 = a^2 - h^2$

- Equation 4 $b^2 - 2bx + x^2 = a^2 - h^2$

- Equation 2 $$x^2 = c^2 - h^2$$
- Equation 3 $$x = \sqrt{c^2 - h^2}$$
- Equation 4 $$b^2 - 2bx + x^2 = a^2 - h^2$$

Substitute x and x^2 from Equations 3 and 2 into Equation 4 …

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides and rearrange …

$$\frac{\left(b^2 + c^2 - a^2\right)^2}{4b^2} = c^2 - h^2$$

$$\frac{\left(b^2+c^2-a^2\right)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{\left(b^2+c^2-a^2\right)^2}{4b^2}$$

$$h^2 = \frac{4b^2c^2-\left(b^2+c^2-a^2\right)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2-\left(b^2+c^2-a^2\right)^2}{4b^2}$$

$$h^2 = \frac{\left[2bc+\left(b^2+c^2-a^2\right)\right]\cdot\left[2bc-\left(b^2+c^2-a^2\right)\right]}{4b^2}$$

$$h^2 = \frac{\left[2bc+\left(b^2+c^2-a^2\right)\right]\cdot\left[2bc-\left(b^2+c^2-a^2\right)\right]}{4b^2}$$

$$h^2 = \frac{\left[2bc+b^2+c^2-a^2\right]\cdot\left[2bc-b^2-c^2+a^2\right]}{4b^2}$$

$$h^2 = \frac{\left[\left(b^2+2bc+c^2\right)-a^2\right]\cdot\left[a^2-\left(b^2-2bc+c^2\right)\right]}{4b^2}$$

$$h^2 = \frac{\left[(b+c)^2-a^2\right]\cdot\left[a^2-(b-c)^2\right]}{4b^2}$$

$$h^2 = \frac{[(b+c)+a]\cdot[(b+c)-a]\cdot[a+(b-c)]\cdot[a-(b-c)]}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}{4b^2}$$

Since … $P = a + b + c$

$$h^2 = \frac{P(P-2a)(P-2b)(P-2c)}{4b^2}$$

- Equation 5 $$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

- Equation 1 $$Area = \frac{b \cdot h}{2}$$
- Equation 5 $$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

Substitute h from Equation 5 into Equation 1 …

$$Area = \frac{1}{2} b \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

$$Area = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{4}$$

$$Area = \sqrt{\frac{P(P-2a)(P-2b)(P-2c)}{16}}$$

$$Area = \sqrt{\frac{P(P-2a)(P-2b)(P-2c)}{16}}$$

$$Area = \sqrt{\left(\frac{P}{2}\right)\left(\frac{P-2a}{2}\right)\left(\frac{P-2b}{2}\right)\left(\frac{P-2c}{2}\right)}$$

$$Area = \sqrt{\frac{P}{2}\left(\frac{P}{2}-a\right)\left(\frac{P}{2}-b\right)\left(\frac{P}{2}-c\right)}$$

Since … $s = P/2 = (a+b+c)/2$

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Heron's Formula

Given the three sides of a triangle (a, b, and c) …

The area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where the semi-perimeter is:

$$s = (a+b+c)/2$$

Spherical Trigonometry

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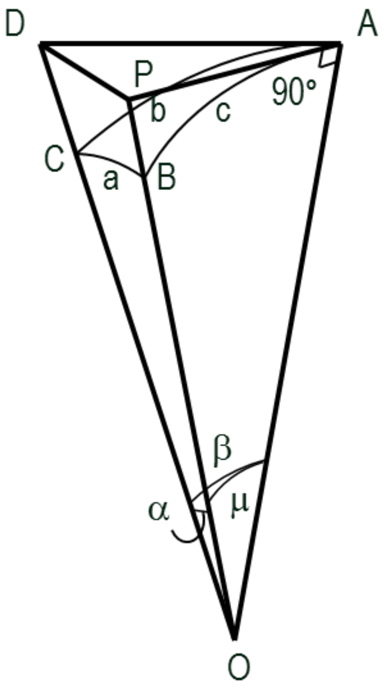
The Spherical Trig <u>*“ LAWS ”*</u>

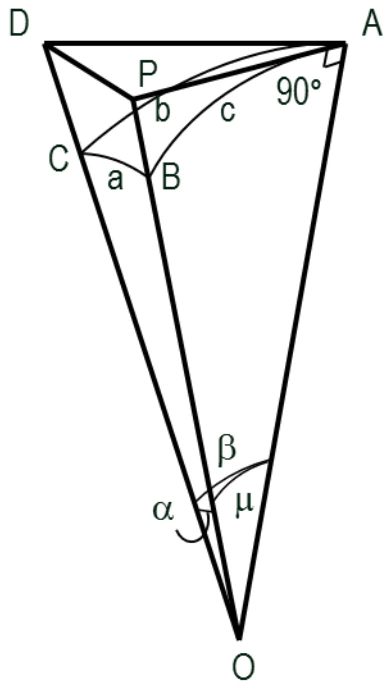
- ***<u>Law of Cosines</u>***
- ***<u>Law of Sines</u>***
- ***<u>Law of Cotangents</u>***
- ***<u>Spherical Excess</u>***



Basic Assumptions

Sides AD and AP are tangent at A such that …

- DAP = spherical angle A
- DAO = 90°
- PAO = 90°



Plane Trig Formulae

In plane triangle DPO …

- **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha$$

In plane triangle DPA …

- **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In plane triangle ADO …

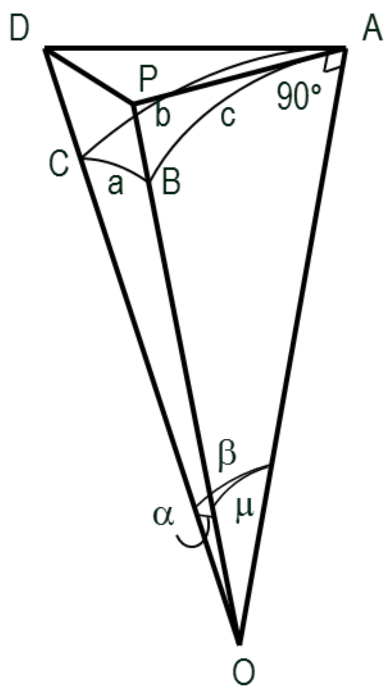
- **Equation 3**

$$DO^2 = AD^2 + AO^2$$

In plane triangle APO …

- **Equation 4**

$$PO^2 = AP^2 + AO^2$$

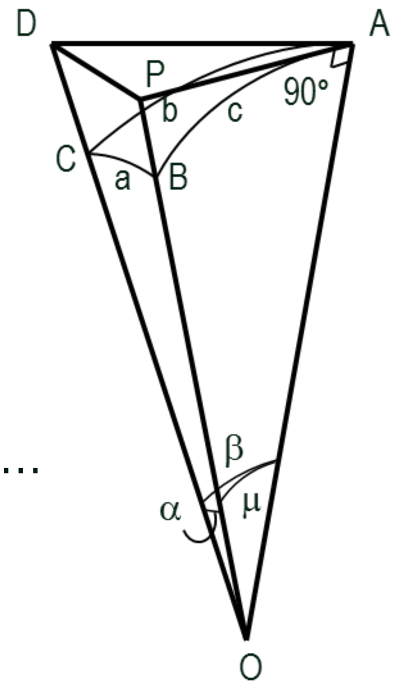


Equation 1

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha$$

Equation 2

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$



Equate the right sides of equations 1 and 2 and rearrange …

$$DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos\alpha =$$
$$AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 5

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 3

$$DO^2 = AD^2 + AO^2$$

Equation 4

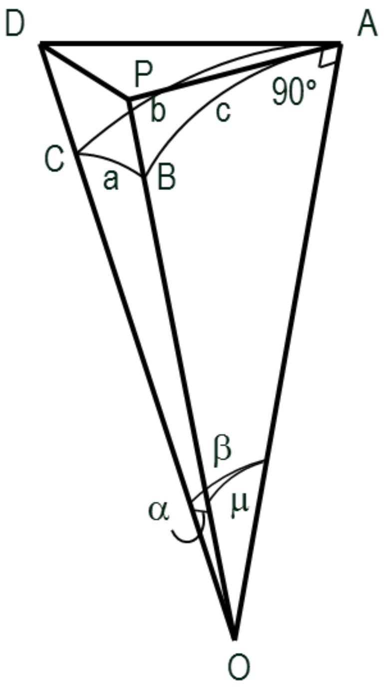
$$PO^2 = AP^2 + AO^2$$

Add equations 3 and 4 and rearrange …

$$DO^2 + PO^2 = AD^2 + AP^2 + AO^2 + AO^2$$

Equation 6

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$

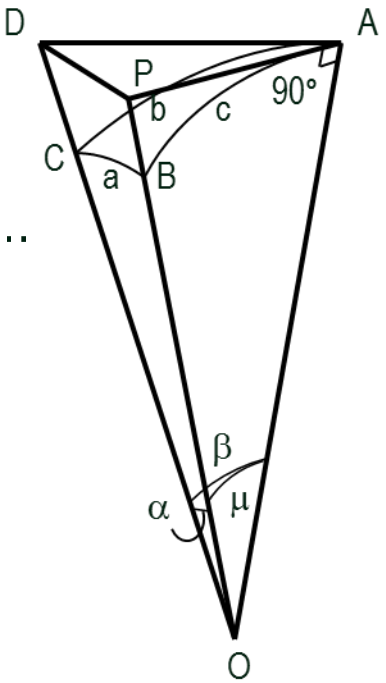


- **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

- **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$



Equate the right sides of equations 5 and 6 and rearrange …

$$2 \cdot DO \cdot PO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A = 2 \cdot AO^2$$

$$DO \cdot PO \cdot \cos\alpha = AO \cdot AO + AD \cdot AP \cdot \cos A$$

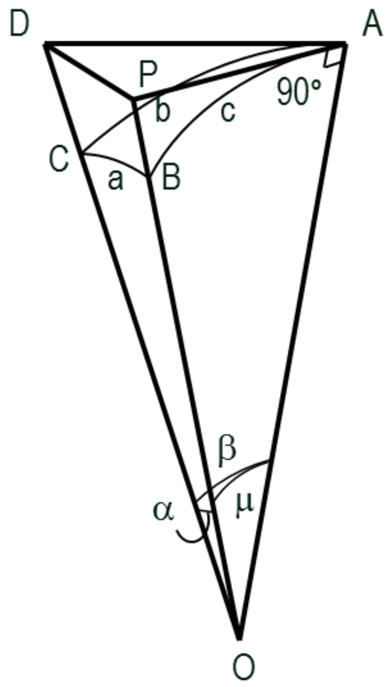
$$\cos\alpha = \frac{AO \cdot AO}{DO \cdot PO} + \frac{AD \cdot AP}{DO \cdot PO}\cos A$$

$$\cos\alpha = \cos\beta \cdot \cos\mu + \sin\beta \cdot \sin\mu \cdot \cos A$$

“LAW of COSINES”

$$\cos\alpha = \cos\beta \cdot \cos\mu + \sin\beta \cdot \sin\mu \cdot \cos A$$

Note … $\alpha = a \quad \beta = b \quad \mu = c$



“LAW of COSINES”

- **Equation 7**

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

- **Equation 8**

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

- **Equation 9**

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$

Rearrange equations 7, 8 and 9 …

- **Equation 10** (from equation 7)

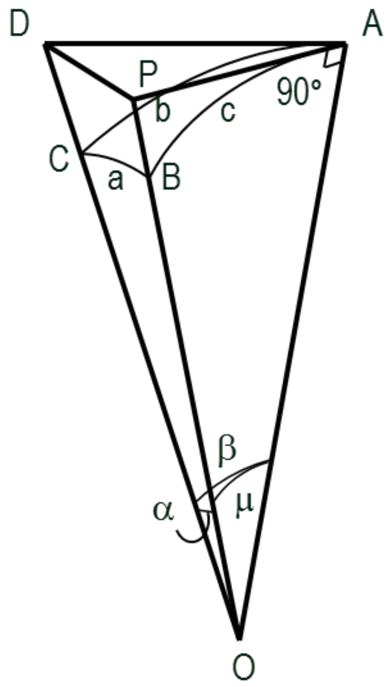
$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

- **Equation 11** (from equation 8)

$$\cos b - \cos a \cdot \cos c = \sin a \cdot \sin c \cdot \cos B$$

- **Equation 12** (from equation 9)

$$\cos c - \cos a \cdot \cos b = \sin a \cdot \sin b \cdot \cos C$$



TERRAMETRA

Equation 10 (from equation 7)

$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

Square both sides of equation 10

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \sin^2 b \cdot \sin^2 c \cdot \cos^2 A$$

Apply the trig identity … $\cos^2 \theta = 1 - \sin^2 \theta$

$$(1 - \sin^2 a) - 2\ \cos a\ \cos b\ \cos c + (1 - \sin^2 b)(1 - \sin^2 c) = \sin^2 b\ \sin^2 c\ (1 - \sin^2 A)$$

$$1 - \sin^2 a - 2\ \cos a\ \cos b\ \cos c + 1 - \sin^2 b - \sin^2 c + \sin^2 b\ \sin^2 c =$$

$$\sin^2 b\ \sin^2 c - \sin^2 b\ \sin^2 c\ \sin^2 A$$

Equation 13 (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

"LAW of SINES"

- **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

- **Equation 14** (from equation 11)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B$$

- **Equation 15** (from equation 12)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

Equate the right sides of equations 13, 14 and 15 and rearrange …

$$-\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

$$\sin b \cdot \sin c \cdot \sin A = \sin a \cdot \sin c \cdot \sin B = \sin a \cdot \sin b \cdot \sin C$$

"LAW of SINES"

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

TERRAMETRA

- **Equation 16** (Law of Sines)

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

- **Equation 17** (Law of Cosines)

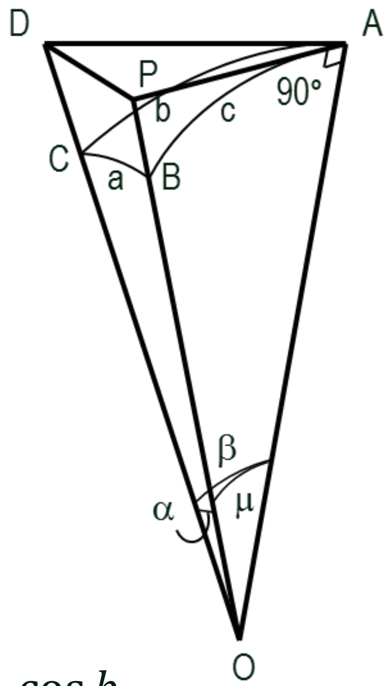
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

- **Equation 18** (Law of Cosines)

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

- **Equation 19** (Law of Cosines)

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$



Substitute the right side of equation 18 into equation 17 for $\cos b$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity … $\sin^2 \theta = 1 - \cos^2 \theta$

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

- **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

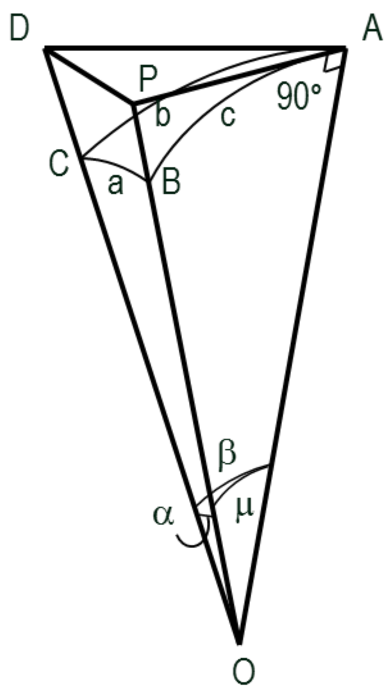
"LAW of COTANGENTS"

- **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

- **Equation 16**

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$



Divide equation 16 into equation 20 and rearrange …

$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

"LAW of COTANGENTS"

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

… or …

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$

Spherical Excess

"SPHERICAL EXCESS"

$$E = A + B + C - 180°$$

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

… where …

$$s = \frac{a + b + c}{2}$$