



TERRAMETRA

PLANE TRIGONOMETRY

Terrametra Resources

Lynn Patten



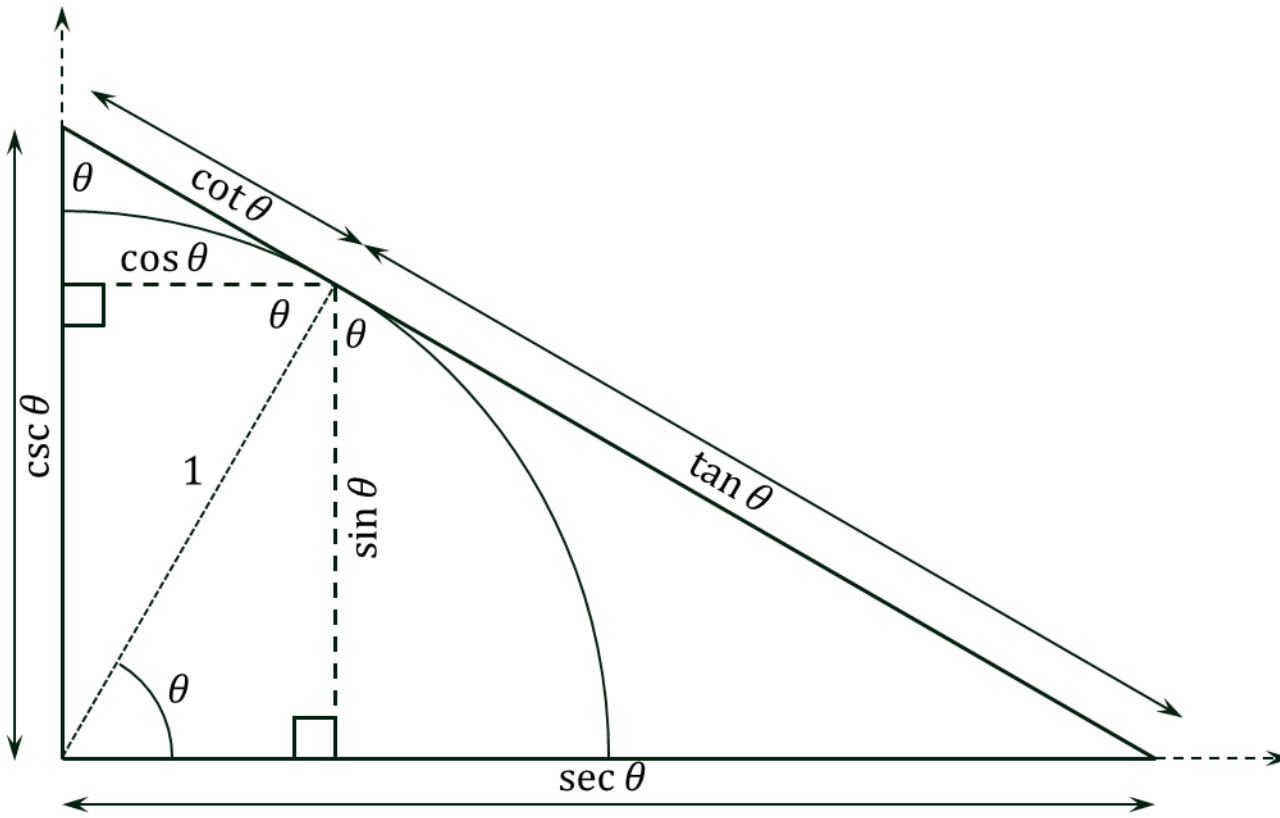
Trigonometric Functions

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Trigonometric Functions



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

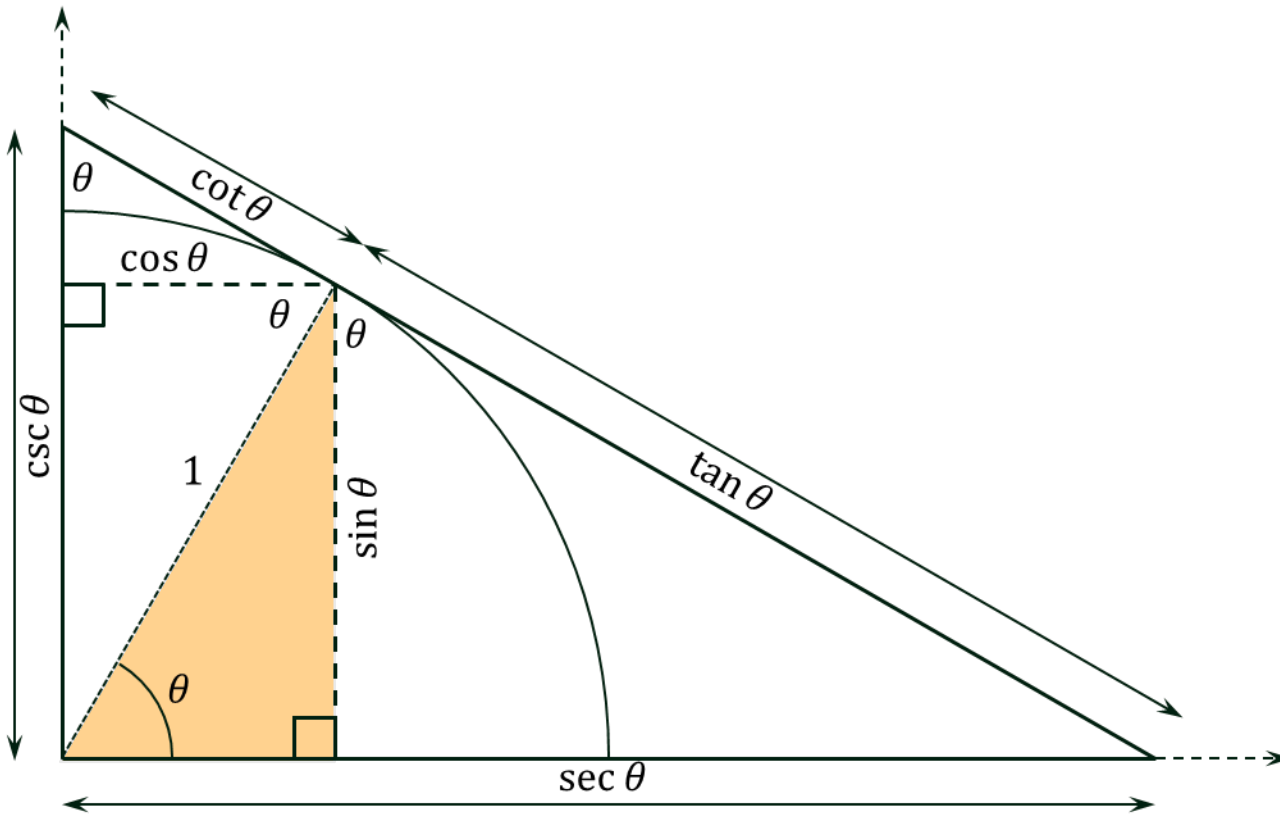
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



SIN Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

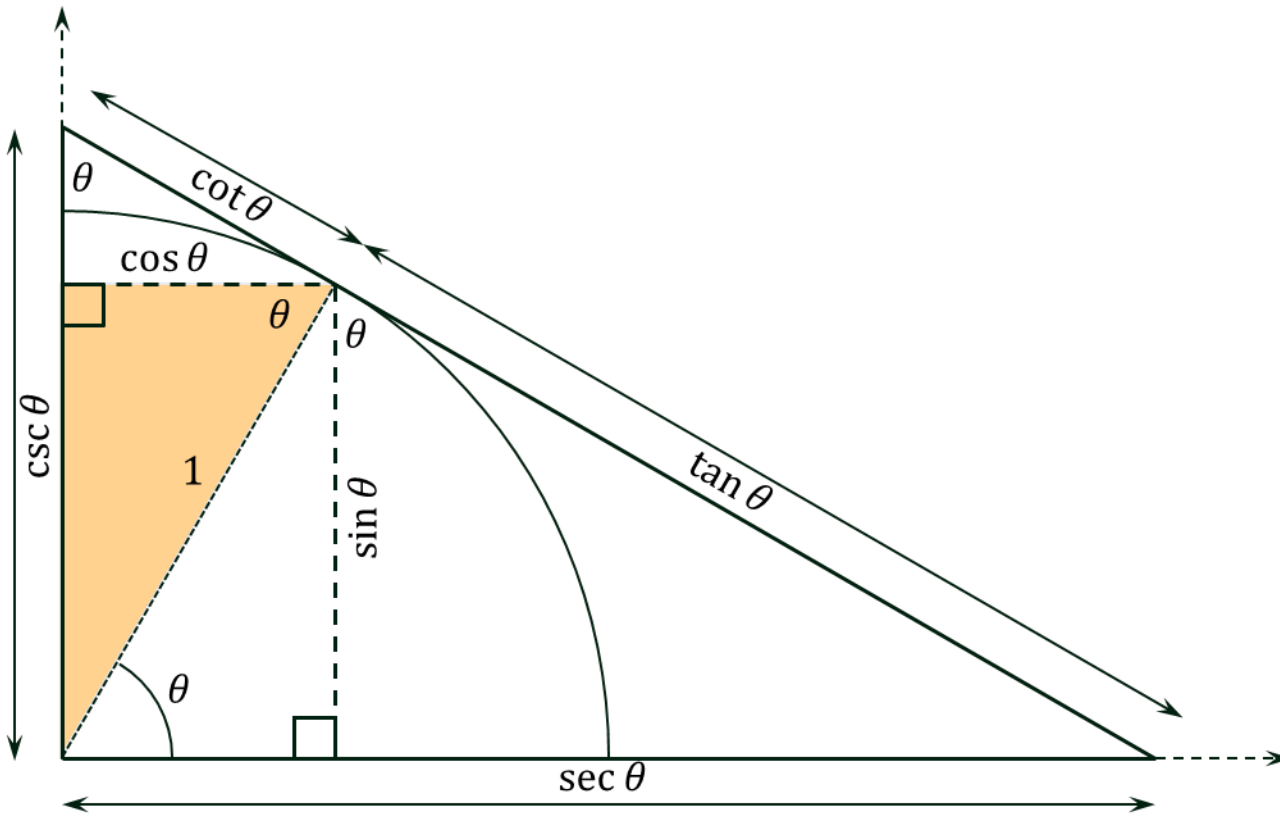
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



COS Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

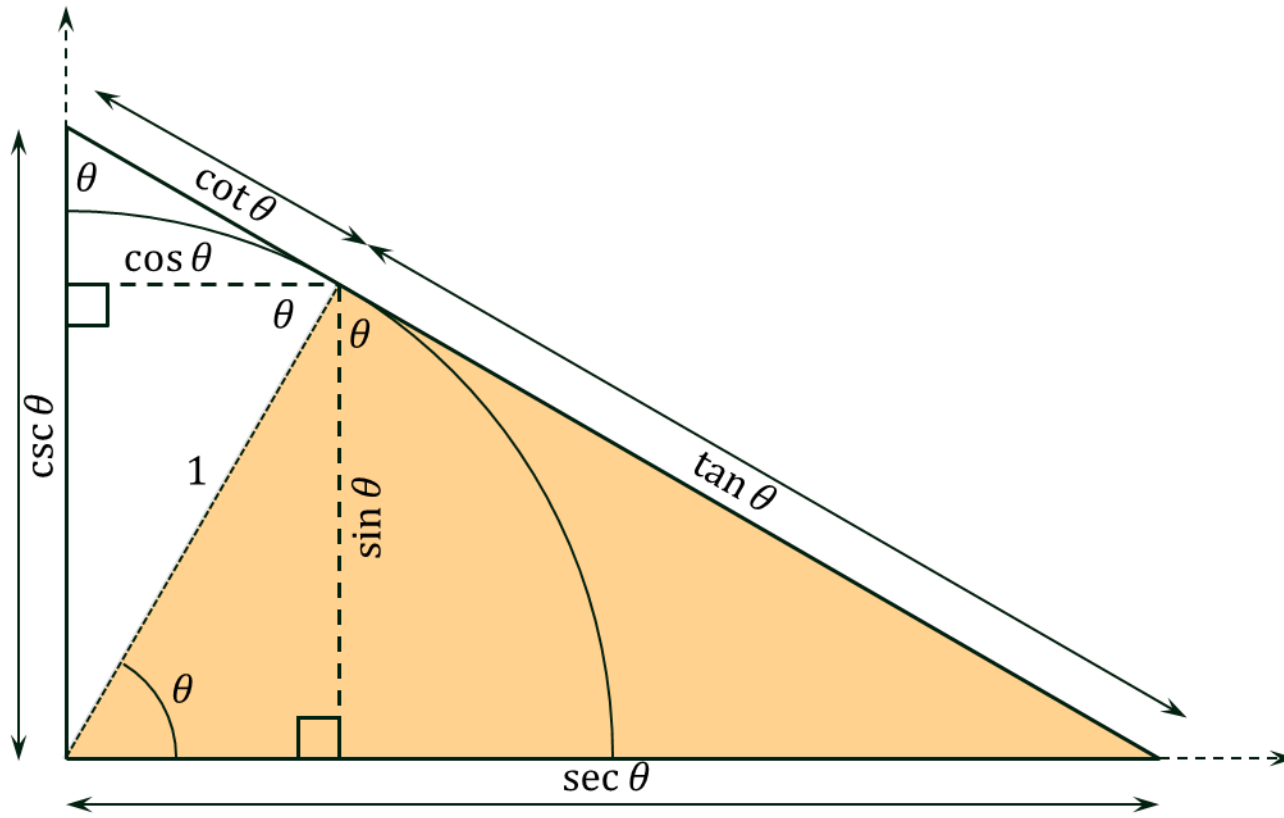
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



TAN Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

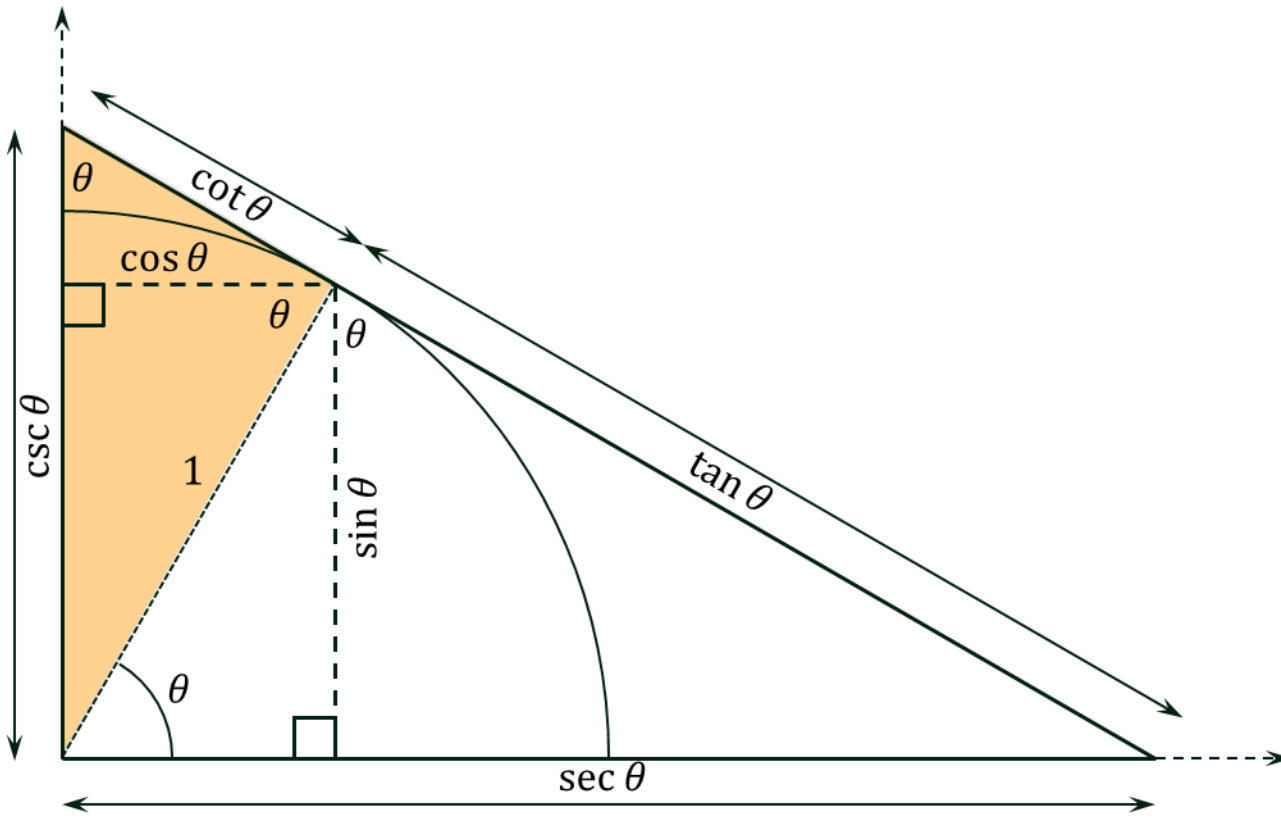
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



COT Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

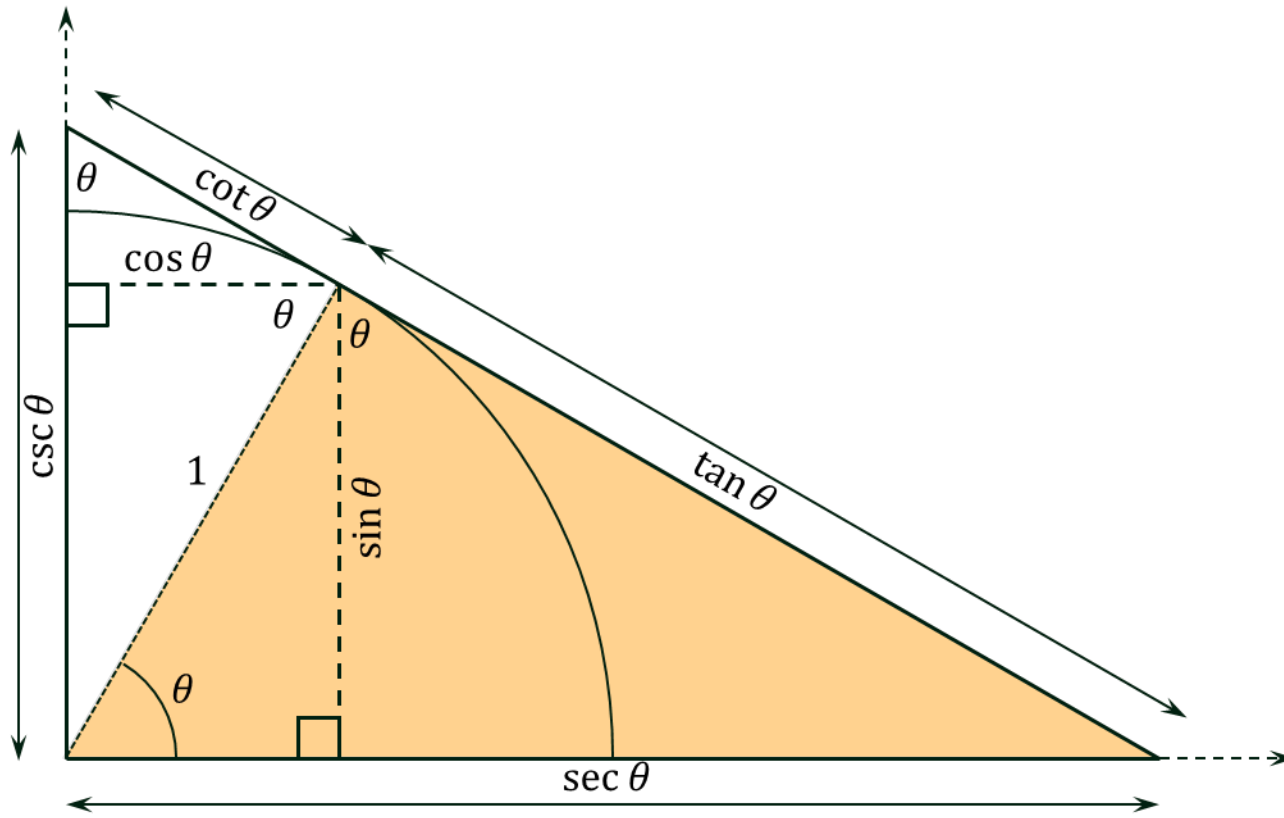
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



SEC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

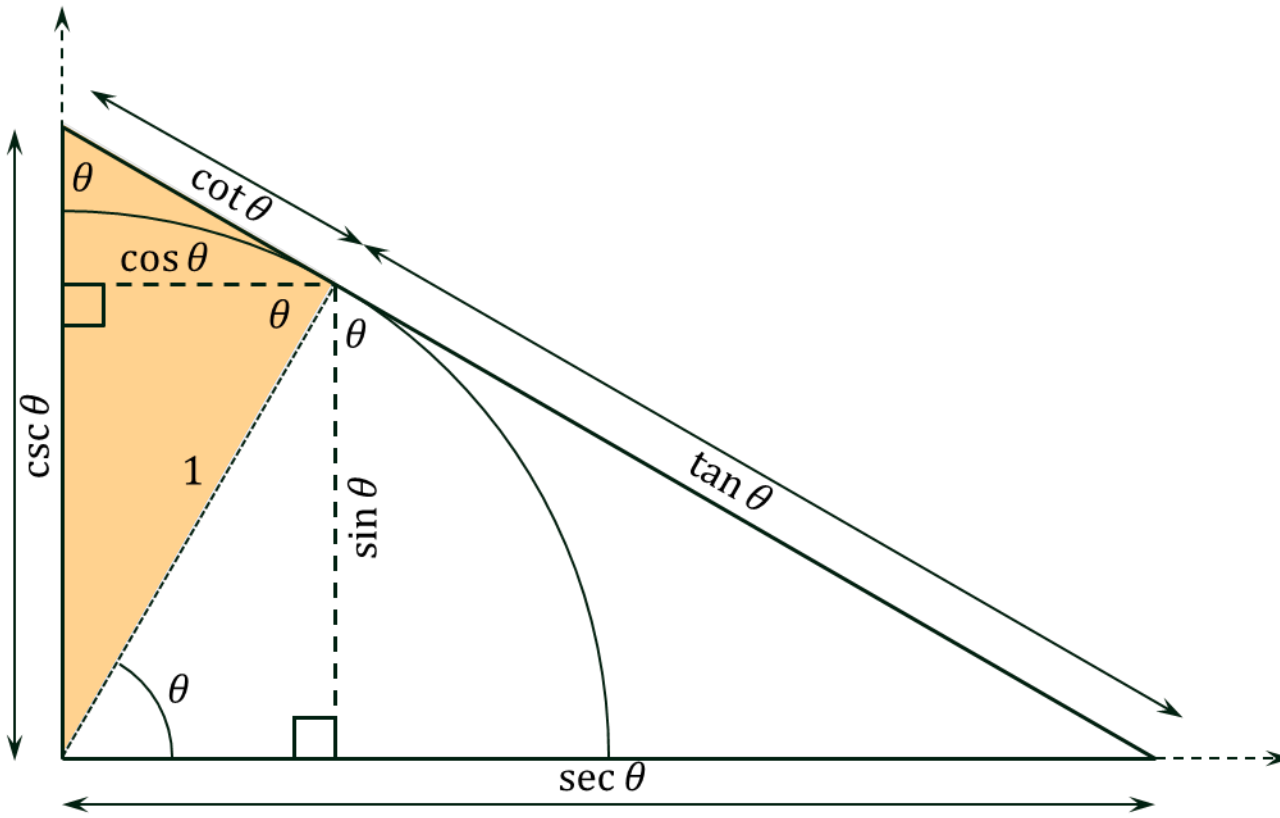
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



CSC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



Trigonometric Identities

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Reciprocal Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$



Ratio Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$$

$$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$$



Pythagorean Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$



Symmetry Identities

The graph of $\sin(\theta)$ is symmetric about the origin,
therefore...

$$\sin(-\theta) = -\sin(\theta)$$

The graph of $\cos(\theta)$ is symmetric about the y-axis,
therefore...

$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin,
therefore...

$$\tan(-\theta) = -\tan(\theta)$$



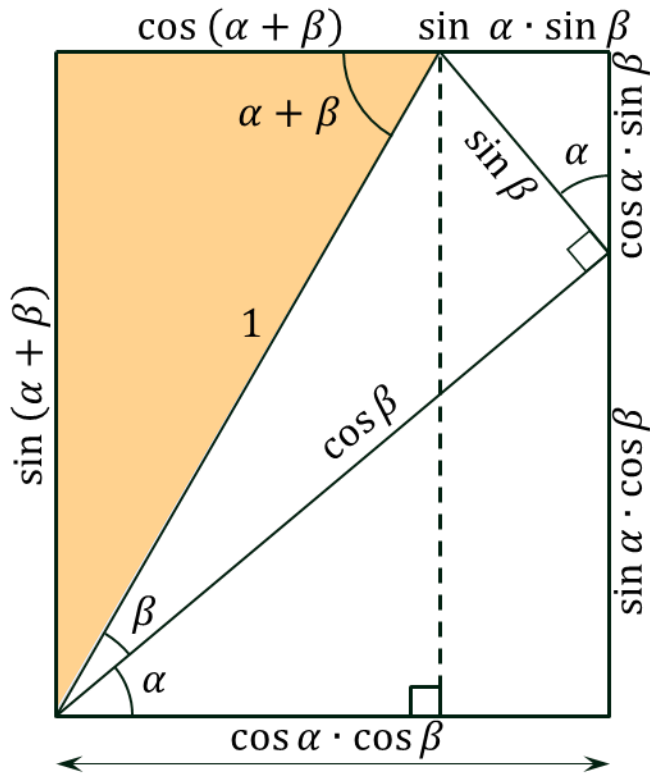
Derived Trigonometric Identities

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SIN of Sums (and Differences)



$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(-\beta) = -\sin \beta$$

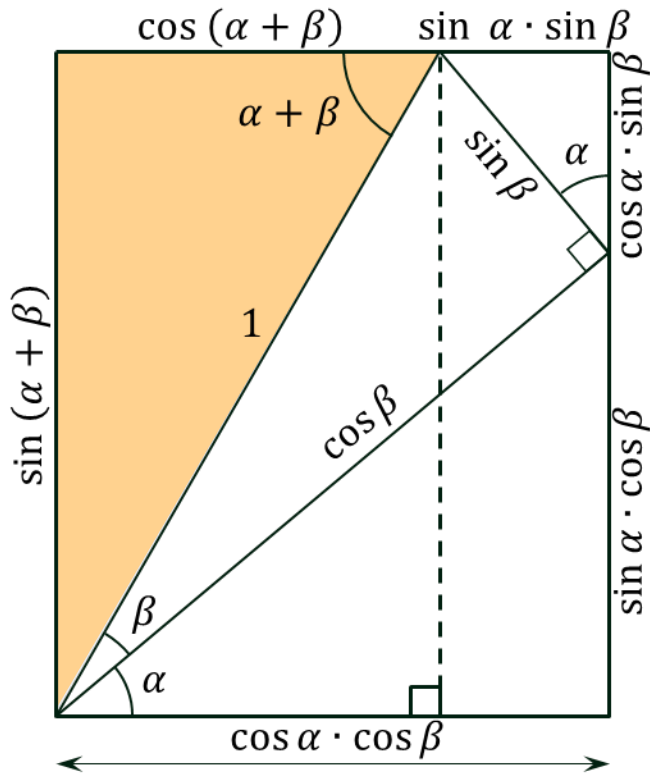
$$\cos(-\beta) = \cos \beta$$

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\ &= \sin \alpha \cdot \cos(-\beta) + \cos \alpha \cdot \sin(-\beta) \end{aligned}$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$



COS of Sums (and Differences)



$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\sin(-\beta) = -\sin \beta$$

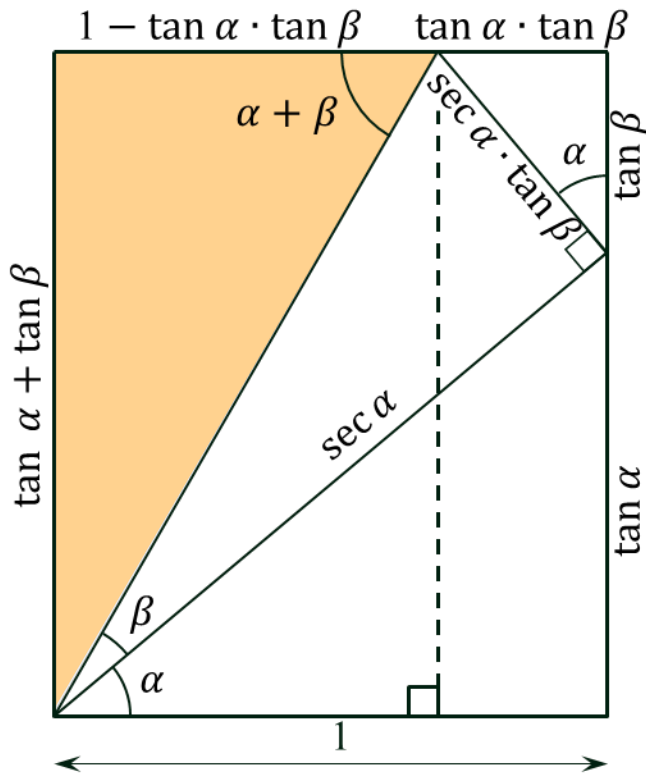
$$\cos(-\beta) = \cos \beta$$

$$\begin{aligned} \cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\ &= \cos \alpha \cdot \cos(-\beta) - \sin \alpha \cdot \sin(-\beta) \end{aligned}$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$



TAN of Sums (and Differences)



$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$\tan(-\beta) = -\tan \beta$$

$$\tan(\alpha - \beta) = \tan(\alpha + (-\beta))$$

$$= \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \cdot \tan(-\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}$$



Double Angle Formulae

$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(2\theta) = \sin(\theta + \theta) = \sin \theta \cdot \cos \theta + \cos \theta \cdot \sin \theta$$

$$\sin(2\theta) = 2 \cdot \sin \theta \cdot \cos \theta$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \cdot \sin^2 \theta$$

$$\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cdot \cos^2 \theta - 1$$



Double Angle Formulae

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \cdot \tan \theta}$$

$$\tan(2\theta) = \frac{2 \cdot \tan \theta}{1 - \tan^2 \theta} \cdot \left(\frac{\cos^2 \theta}{\cos^2 \theta} \right) =$$

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2 \cdot \sin \theta \cdot \cos \theta}{\cos^2 \theta - \sin^2 \theta}$$



Half Angle Formulae

$$\cos(2\theta) = 1 - 2 \cdot \sin^2\theta$$

$$2 \cdot \sin^2\theta = 1 - \cos(2\theta)$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin \theta = \pm \sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\sin \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos(\phi)}{2}}$$

$$\cos(2\theta) = 2 \cdot \cos^2\theta - 1$$

$$2 \cdot \cos^2\theta = 1 + \cos(2\theta)$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\cos \theta = \pm \sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\cos \frac{\phi}{2} = \pm \sqrt{\frac{1 + \cos(\phi)}{2}}$$

$$\frac{\phi}{2} = \theta$$



Half Angle Formulae

$$\tan \frac{\phi}{2} = \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} = \frac{\pm \sqrt{\frac{1 - \cos(\phi)}{2}}}{\pm \sqrt{\frac{1 + \cos(\phi)}{2}}}$$

$$\tan \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos(\phi)}{1 + \cos(\phi)}}$$

$$\tan \frac{\phi}{2} = \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} \cdot \frac{\sqrt{1 + \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} = \frac{\sqrt{1 - \cos^2(\phi)}}{1 + \cos(\phi)}$$

$$= \frac{\sin(\phi)}{1 + \cos(\phi)}$$

$$\tan \frac{\phi}{2} = \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} \cdot \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 - \cos(\phi)}} = \frac{1 - \cos(\phi)}{\sqrt{1 - \cos^2(\phi)}}$$

$$= \frac{1 - \cos(\phi)}{\sin(\phi)}$$



Trigonometric Laws

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“LAW of SINES”

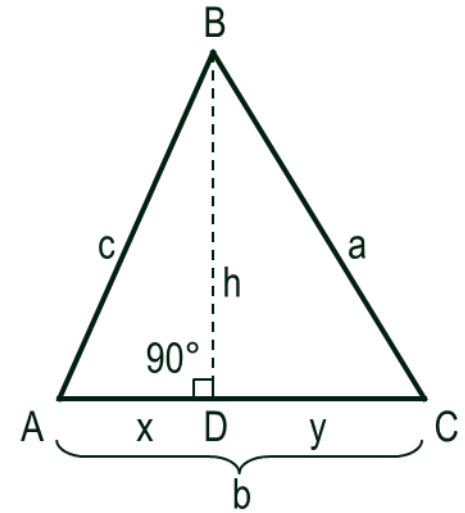
✓ **Equation 1** $h = c \cdot \sin A$

✓ **Equation 2** $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange ...

✓ **Equation 3** $\frac{a}{\sin A} = \frac{c}{\sin C}$

Similarly: $\frac{a}{\sin A} = \frac{b}{\sin B}$ and $\frac{b}{\sin B} = \frac{c}{\sin C}$



“LAW of SINES”

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}} \quad \dots \text{ or } \dots \quad \boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$



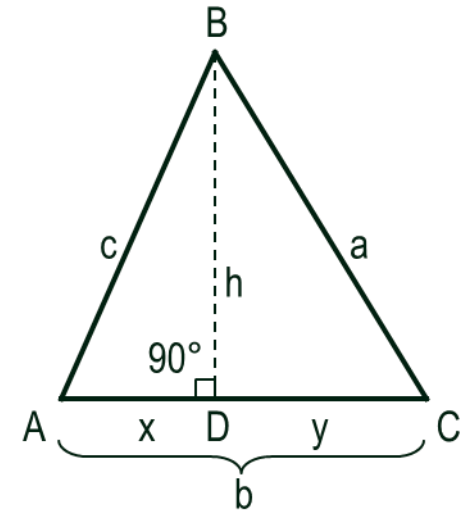
“LAW of COSINES”

- ✔ **Equation 1** $h^2 = c^2 - x^2$
- ✔ **Equation 2** $h^2 = a^2 - y^2 = a^2 - (b - x)^2$

Equate the right sides of equations 1 and 2 and rearrange ...

$$a^2 - (b - x)^2 = c^2 - x^2$$

- ✔ **Equation 3** $a^2 = b^2 + c^2 - 2bx$
- ✔ **Equation 4** $x = c \cos A$



Substitute for x from equation 4 into equation 3...

“LAW of COSINES”

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Similarly: {

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



Plane Triangle Areas

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Plane Triangle Areas

$$\text{Area } ABD = \text{Area } ABE = \frac{x \cdot h}{2}$$

$$\text{Area } CBD = \text{Area } CBF = \frac{y \cdot h}{2}$$

✓ Area of Triangle ABC...

$$\text{Area} = \frac{b \cdot h}{2}$$

$$h = a \sin C$$

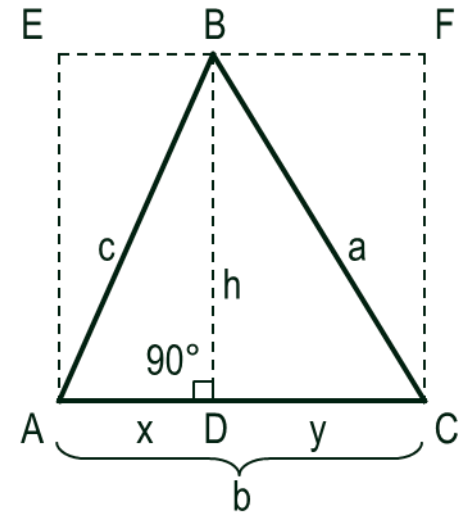
✓ Area of Triangle ABC...

$$\text{Area} = \frac{a \cdot b \cdot \sin C}{2}$$

$$b = \frac{a \cdot \sin B}{\sin A}$$

✓ Area of Triangle ABC...

$$\text{Area} = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$





Heron's Formula

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▼ Equation 1

$$\text{Area} = \frac{b \cdot h}{2}$$

From triangle ABD...

$$x^2 + h^2 = c^2$$

▼ Equation 2

$$x^2 = c^2 - h^2$$

▼ Equation 3

$$x = \sqrt{c^2 - h^2}$$

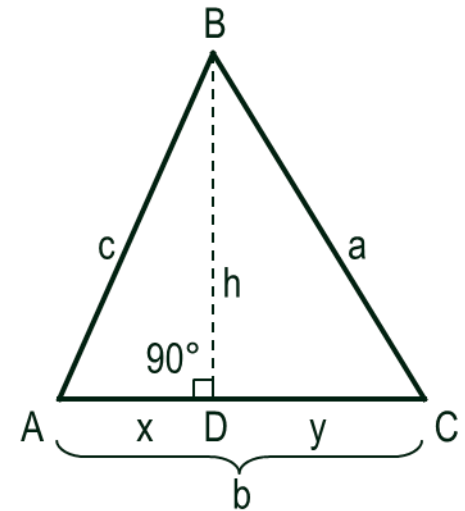
From triangle CBD...

$$(b - x)^2 + h^2 = a^2$$

$$(b - x)^2 = a^2 - h^2$$

▼ Equation 4

$$b^2 - 2bx + x^2 = a^2 - h^2$$





✓ Equation 2

$$x^2 = c^2 - h^2$$

✓ Equation 3

$$x = \sqrt{c^2 - h^2}$$

✓ Equation 4

$$b^2 - 2bx + x^2 = a^2 - h^2$$

Substitute x and x^2 from Equations 3 and 2 into Equation 4 ...

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides and rearrange ...

$$\frac{(b^2 + c^2 - a^2)^2}{4b^2} = c^2 - h^2$$



$$\frac{(b^2+c^2-a^2)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{4b^2c^2 - (b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2 - (b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{[2bc + (b^2+c^2-a^2)] \cdot [2bc - (b^2+c^2-a^2)]}{4b^2}$$



$$h^2 = \frac{[2bc + (b^2 + c^2 - a^2)] \cdot [2bc - (b^2 + c^2 - a^2)]}{4b^2}$$

$$h^2 = \frac{[2bc + b^2 + c^2 - a^2] \cdot [2bc - b^2 - c^2 + a^2]}{4b^2}$$

$$h^2 = \frac{[(b^2 + 2bc + c^2) - a^2] \cdot [a^2 - (b^2 - 2bc + c^2)]}{4b^2}$$

$$h^2 = \frac{[(b+c)^2 - a^2] \cdot [a^2 - (b-c)^2]}{4b^2}$$

$$h^2 = \frac{[(b+c)+a] \cdot [(b+c)-a] \cdot [a+(b-c)] \cdot [a-(b-c)]}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$



$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$
$$h^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2}$$
$$h^2 = \frac{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}{4b^2}$$

Since ...

$$P = a + b + c$$

$$h^2 = \frac{P(P-2a)(P-2b)(P-2c)}{4b^2}$$

▼ Equation 5

$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$



✓ Equation 1

$$Area = \frac{b \cdot h}{2}$$

✓ Equation 5

$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

Substitute h from Equation 5 into Equation 1 ...

$$Area = \frac{1}{2} b \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

$$Area = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{4}$$

$$Area = \sqrt{\frac{P(P-2a)(P-2b)(P-2c)}{16}}$$



$$Area = \sqrt{\frac{P(P-2a)(P-2b)(P-2c)}{16}}$$

$$Area = \sqrt{\left(\frac{P}{2}\right) \left(\frac{P-2a}{2}\right) \left(\frac{P-2b}{2}\right) \left(\frac{P-2c}{2}\right)}$$

$$Area = \sqrt{\frac{P}{2} \left(\frac{P}{2} - a\right) \left(\frac{P}{2} - b\right) \left(\frac{P}{2} - c\right)}$$

Since ...

$$s = P/2 = (a + b + c)/2$$

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$



Heron's Formula

Given the three sides of a triangle (a, b, and c) ...

The area of the triangle is:

$$Area = \sqrt{s(s - a)(s - b)(s - c)}$$

Where the semi-perimeter is:

$$s = (a + b + c)/2$$